Kill's Euclid's Elements
of
Geometry
1767

Dr. KEILL's

PREFACE.

A YOUNG Mathematician may be surprized to see the old obsolete Elements of Euclid appear afresh in Print; and that, too, after so many new Elements of Geometry as have been lately published; especially those who gave us the Elements of Geometry, in a new Manner, would have us believe they have detected a great many Faults in Euclid. These acute Philosophers pretend to have discovered, that Euclid's Definitions are not perspicuous enough; that his Demonstrations are scarcely evident; that his whole Elements are ill-disposed; and that they have found out innumerable Falsities in them, which had lain hid to their Times.

But, by their Leave, I make bold to affirm, that they carp at Euclid undeservedly: for his Definitions are distinct and clear, as being taken from first Principles, and our most easy and simple Conceptions; and his Demonstrations elegant, perspicuous, and concise, carrying with them such Evidence, and so much Strength of Reason, that I am easily induced to believe, that the Obscurity Sciolists so often accuse Euclid with, is rather to be attributed to their own per-

finition. And why might not the Author of the Elements give what Names he thought fit to Quantities, having such Requisites? Surely he might use his own Liberty, and accordingly has called them Proportionals.

But it may be proper here to examine the Method whereby they endeavour to demonstrate that Property: Which is by first assuming a certain Affection, agreeing only to one kind of Proportionals, viz. Commensurables; and thence, by a long Circuit, and a perplexed Series of Conclusions, do deduce that universal Property of Proportionals which Euclid *affirms; a Procedure foreign enough to the just Methods and Rules of Reasoning. They would certainly have done much better, if they had first laid down that universal Property by* Euclid, *and thence have deduced that particular Property agreeing to only one Species of Proportionals. But, rejecting this Method, they have taken the Liberty of adding their Demonstration to this Definition of the fifth Book. Those who have a mind to see a farther Defence of* Euclid, *may consult the Mathematical Lectures of the learned Dr.* Barrow.

As I have happened to mention this great Geometrician, I must not pass by the Elements published by him, wherein, generally, he has retained the Constructions and Demonstrations of Euclid *himself, not having omitted so much as one Proposition. Hence, his Demonstrations become more strong and nervous, his Constructions more neat and elegant,*

elegant, and the Genius of the antient Geometricians more conspicuous, than is usually found in other Books of this Kind. To this are his added several Corollaries and Scholia, which serve not only to shorten the Demonstration of what follows, but are likewise of use in other Matters.

Notwithstanding this, Barrow's *Demonstrations are so very short, and are involved in so many Notes and Symbols, that they are rendered obscure and difficult to one not versed in Geometry. There, many Propositions, which appear conspicuous in reading* Euclid *himself, are made knotty, and scarcely intelligible to Learners, by his Algebraical Way of Demonstration; as is, for Example,* Prop. 13. Book I. *And the Demonstrations which he lays down in* Book II. *are still more difficult:* Euclid *himself has done much better, in shewing their Evidence by the Contemplations of Figures, as in Geometry should always be done. The Eelements of all sciences ought to be handled after the most simple Method, and not to be involved in Symbols, Notes, or obscure Principles, taken elsewhere.*

As Barrow's *Elements are too short, so are those of* Clavius *too prolix, abounding in superfluous Scholiums and Comments: For, in my Opinion,* Euclid *is not so obscure as to want such a Number of Notes, neither do I doubt, but a Learner will find* Euclid *much easier than any of his Commentators. As too much Brevity in Geometrical Demon-*

ſtrations begets Obſcurity, ſo too much Prolixity produces Tedioufneſs and Confuſion.

On theſe Accounts, principally, it was that I undertook to publiſh the firſt ſix Books of Euclid, *with the* 11*th and* 12*th, according to* Commandinus's *Edition; the reſt I forbore, becauſe thoſe, firſt-mentioned, are ſufficient for underſtanding of moſt Parts of the Mathematics now ſtudied.*

Farther, for the Uſe of thoſe who are deſirous to apply the Elements of Geometry to Uſes in Life, we have added a Compendium of Plane *and* Spherical Trigonometry; *by Means whereof, Geometrical Magnitudes are meaſured, and their Dimenſion expreſſed in Numbers.*

J. KEILL.

Mr. *CUNN*'s

PREFACE,

SHEWING,

The USEFULNESS *and* EXCELLENCY *of this* WORK.

DR. *KEILL*, in his Preface, hath ſufficiently declared how much eaſier, plainer, and more elegant, the Elements of Geometry written by *Euclid* are, than thoſe written by others; and that the Elements themſelves are fitter for a Learner, than thoſe publiſhed by ſuch as have pretended to comment on, ſymbolize, or tranſpoſe, any of his Demonſtrations of ſuch Propoſitions as they intend to treat of. Then how muſt a Geometrician be amazed, when he meets with a Tract * of the 1ſt, 2d, 3d, 4th, 5th, 6th, 11th, and 12th *Books* of the Elements, in which are omitted the Demonſtrations of all the Propoſitions of that moſt noble univerſal Matheſis, the 5th; on which the 6th, 11th, and 12th, ſo much depend, that the Demonſtration of not ſo much as one Propoſition, in them, can be obtained without thoſe in the fifth!

* *Vide* the laſt Edition of the *Engliſh* Tacquet.

Tho

The 7th, 8th, and 9th Books, treat of ſuch Properties of Numbers which are neceſſary for the Demonſtration of the 10th, which treats of Incommenſurables; and the 13th, 14th, and 15th, of the five *Platonic* Bodies. But though the Doctrine of Incommenſurables, becauſe expounded in one and the ſame Plane, as the firſt ſix Elements were, claimed, by a Right Order, to be handled before Planes interſected by Planes, or the more compounded Doctrine of Solids; and the Properties of Numbers were neceſſary to the Reaſoning about Incommenſurables; yet, becauſe only one Propoſition of theſe four Book, *viz.* the 1ſt of the 10th, is quoted in the 11th and 12th Books; and that only twice, *viz.* in the Demonſtration of the 2d and 16th of the 12th; and that 1ſt Propoſition of the 19th is ſupplied by a *Lemma* in the 12th; and becauſe the 7th, 8th, 9th, 10th, 13th, 14th, and 15th Books have not been thought (by our greateſt Maſters) neceſſary to be read by ſuch as deſign to make Natural Philoſophy their Study, or by ſuch as would apply Geometry to practical Affairs; Dr. *Keill*, in his Edition, gave us only theſe eight Books, *viz.* the firſt ſix, and the 11th and 12th.

And as he found there was wanting a Treatiſe of theſe Parts of the Elements, as they were written by *Euclid* himſelf; he

he published his Edition without omitting any of *Euclid*'s Demonstrations, except two; one of which was a second Demonstration of the 9th Proposition of the third Book; and the other a Demonstration of that Property of Proportionals called *Conversions* (contained in a *Corollary* to the 19th Proposition of the fifth Book;) where, instead of *Euclid*'s Demonstration, which is universal, most Authors have given us only particular ones of their own. The first of these, which was omitted, is here supplied: And that which was corrupted is here restored *.

And since several Persons, to whom the Elements of Geometry are of vast Use, either are not so sufficiently skilled in, or perhaps have not Leisure, or are not willing to take the Trouble, to read the *Latin*; and since this Treatise was not before in *English*, nor any other which may properly be said to contain the Demonstrations laid down by *Euclid* himself; I do not doubt but the Publication of this Edition will be acceptable, as well as serviceable.

Such Errors, either typographical, or in the Schemes, which were taken Notice of in the *Latin* Edition, are corrected in this.

* *Vide Page* 55. 107. of *Euclid*'s Works, published by Dr. *Gregory*.

As

As to the Trigonometrical Tract, annexed to these Elements, I find our Author, as well as Dr. *Harris*, Mr. *Caswell*, Mr. *Haynes*, and others of the Trigonometrical Writers, is mistaken in some of the Solutions.

That the common Solution of the 12th *Case* of Oblique Spherics is false, I have demonstrated, and given a true one. See *Page* 318.

In the Solution of our 9th and 10th *Cases*, by our Authors called the 1st and 2d, where are given and sought opposite Parts, not only the afore-mentioned Authors, but all others that I have met with, have told us, that the Solutions are ambiguous; which Doctrine is, indeed, sometimes true, but sometimes false: For sometimes the *Quæsitum* is doubtful, and sometimes not; and when it is not doubtful, it is sometimes greater than 90 Degrees, and sometimes less: And sure I shall commit no Crime, if I affirm, that no Solution can be given without a just Distinction of these Varieties. For the Solution of these *Cases*, see *Pages* 320, 321.

In the Solution of our 3d and 7th *Cases*, in other Authors reckoned the 3d and 4th, where there are given two Sides, and an Angle opposite to one of them, to find the 3d Side, or the Angle opposite to it; all the Writers

Writers of Trigonometry, that I have met with, who have undertaken the Solutions of these two, as well as the two following *Cases*, by letting fall a Perpendicular, which is undoubtedly the shortest and best Method for finding either of these *Quæsita*, have told us, that the $\left\{\begin{array}{l}\text{Sum}\\ \text{Difference}\end{array}\right\}$ of the Vertical Angles, or Bases, shall be the sought Angle or Side, according as the Perpendicular falls $\left\{\begin{array}{l}\text{without ;}\\ \text{within ;}\end{array}\right\}$ which cannot be known, unless the Species of that unknown Angle, which is opposite to a given Side, be first known.

Here they leave us first to calculate that unknown Angle, before we shall know whether we are to take the Sum or the Difference of the vertical Angles or Bases for the sought Angle or Base: And in the Calculation of that Angle have left us in the Dark as to its Species; as appears by the Observations on the two preceding *Cases*.

The Truth is, the *Quæsitum* here, as well as in the two former *Cases*, is sometimes doubtful, and sometimes not; when doubtful, sometimes each Answer is less than 90 Degrees, sometimes each is greater; but sometimes one less, and the other greater, as in the two last-mentioned *Cases*. When it is not doubtful, the *Quæ-*

situm

situm is sometimes greater than 90 Degrees, and sometimes less; all which Distinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, opposite to a given Side; or, which is the same Thing, the Falling of the Perpendicular within or without. For which, see *Pages* 323, 324.

In the Solution of our 1st and 5th *Cases*, called in other Authors the 5th and 6th; where there are given two Angles, and a Side opposite to one of them, to find the third Angle, or the Side opposite to it; they have told us, that the $\left\{\begin{matrix}\text{Sum}\\\text{Difference}\end{matrix}\right\}$ of the vertical Angles, or Bases, according as the Perpendicular falls $\left\{\begin{matrix}\text{within,}\\\text{without,}\end{matrix}\right\}$ shall be the sought Angle or Side; and that it is known whether the Perpendicular falls within, or without, by the Affection of the given Angles.

Here they seem to have spoken as tho' the *Quæsitum* was always determined, and never ambiguous; for they have here determined whether the Perpendicular falls within or without, and thereby whether they are to take the Sum or the Difference of the vertical Angles or Bases for the sought Angle or Side.

But,

But, notwithſtanding theſe imaginary Determinations, I affirm, that the *Quæſitum*, here, as in the two *Caſes* laſt-mentioned, is ſometimes ambiguous, and ſometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See the Solutions of theſe two *Caſes* in *Page* 322.

The Determination of the 3d *Caſe* of Oblique Plane Triangles, ſee in *Page* 224.

SAM. CUNN.

Adver-

Advertiſement.

THE Reader is now preſented with a more correct Edition of this Work, than any hitherto extant; for, not only many Typographical Errors had by Degrees crept into it, but there were many Omiſſions and Miſtakes, even in the Firſt Edition, the greater Part of which have been conſtantly adhered to, in the five ſubſequent ones. Upon the Application of the Proprietors for a Reviſion of this Work, the Reviſor was favoured, by Mr. *John Robertſon*, F. R. S. late Maſter of the Royal Mathematical School in *Chriſt's* Hoſpital, with an interleaved Copy of the firſt Edition thereof, in which are a great Number of Additions and Corrections of Mr. *Cunn's* own Hand-writing, deſigned (as may be ſuppoſed) to have been inſerted in a Second Edition; but probably, prevented from ſo being, either by his Death, or ſome other Accident: All theſe Alterations have been carefully made, in this Edition, and ſeveral more Errors, even in that Edition which had eſcaped Mr. *Cunn's* Notice, and have been continued in the following Editions, are in this corrected.

After theſe Amendments had been made, in the printed Copy of the Sixth Edition, the Reviſor carefully peruſed the ſame, and rectified great Numbers of falſe References to the Plates, and ſome Errors in the Plates themſelves (for they are not the ſame with thoſe annexed to the Firſt Edition): But the moſt flagrant Typographical Errors appeared in the Algebraic Series, given in the Treatiſes on Trigonometry and Logarithms, and demonſtrated in the Appendix; for the greateſt Part of theſe were ſo badly diſpoſed, as to be unintelligible, even to thoſe who underſtand the Subject; theſe are here rendered intelligible, and the Whole now is (as the Reviſor apprehends) in ſuch a State, as the ſeveral Authors of the Work and Appendix would have choſe to have put it into, had they been alive ſo to do.

EUCLID's

EUCLID's
ELEMENTS.

BOOK I.

DEFINITIONS.

I. *A POINT is that which hath no Parts or Magnitude.*

II. *A Line is Length, without Breadth.*

III. *The Ends (or Bounds) of a Line are Points.*

IV. *A Right Line is that which lieth evenly between its Points.*

V. *A Superficies is that which hath only Length and Breadth.*

VI. *The Bounds of a Superficies are Lines.*

VII. *A plane Superficies is that which lieth evenly between its Lines.*

VIII. *A plane Angle is the Inclination of two Lines to one another in the same Plane, which touch each other, but do not both lie in the same Right Line.*

IX. *If the Lines containing the Angle be Right ones, then the Angle is called a Right-lined Angle.*

X. *When one Right Line, standing on another Right Line, makes Angles on either Side there-*

 of,

of, equal between themselves, each of these equal Angles is a Right one; and that Right Line, which stands upon the other, is called a Perpendicular to that whereon it stands.

XI. *An Obtuse Angle is that which is greater than a Right one.*

XII. *An Acute Angle is that which is less than a Right one.*

XIII. *A Term (or Bound) is that which is the Extreme of any Thing.*

XIV. *A Figure is that which is contained under one or more Terms.*

XV. *A Circle is a plain Figure, contained under one Line, called the Circumference; to which all Right Lines, drawn from a certain Point within the Figure, are equal.*

XVI. *And that Point is called the Centre of the Circle.*

XVII. *A Diameter of a Circle is a Right Line drawn through the Centre, and terminated on both Sides by the Circumference, and divides the Circle into two equal Parts.*

XVIII. *A Semicircle is a Figure contained under a Diameter, and that Part of the Circumference of a Circle cut off by that Diameter.*

XIX. *A Segment of a Circle is a Figure contained under a Right Line, and Part of the Circumference of the Circle [which is cut off by that Right Line.]*

XX. *Right lined Figures are such as are contained under Right Lines.*

XXI. *Three-sided Figures are such as are contained under three Lines.*

XXII. *Four-sided Figures are such as are contained under four Lines.*

XXIII. *Many-sided Figures are those that are contained under more than four Right Lines.*

XXIV. *Of three-sided Figures that is an Equilateral Triangle, which hath three equal Sides.*

XXV. *That an Isosceles, or Equicrural one, which hath only two Sides equal.*

XXVI. *And a Scalene one, is that, which hath three unequal Sides.*

XXVII. *Also of three-sided Figures, that is a Right-angled Triangle, which hath a Right Angle.*

XXVIII. *That an Obtuse-angled one, which hath an Obtuse Angle.*

XXIX. *And that an Acute-angled one, which hath three Acute Angles.*

XXX. *Of four-sided Figures, that is a Square, whose four Sides are equal, and its Angles all Right ones.*

XXXI. *That an Oblong, or Rectangle, which is longer than broad; but its opposite Sides are equal, and all its Angles Right ones.*

XXXII. *That a Rhombus, which hath four equal Sides, but not Right Angles.*

XXXIII. *That a Rhomboides, whose opposite Sides and Angles only are equal.*

XXXIV. *All Quadrilateral Figures, besides these, are called Trapezia.*

XXXV. *Parallels are such Right Lines, in the same Plane, which, if infinitely produc'd both Ways, would never meet.*

POSTULATES.

I. *GRANT that a Right Line may be drawn from any one Point to another.*

II. *That a finite Right Line may be continued directly forwards.*

III. *And that a Circle may be described about any Centre with any Distance.*

AXIOMS.

I. *THINGS equal to one and the same Thing, are equal to one another.*

II. *If to equal Things are added equal Things, the Wholes will be equal.*

III. *If from equal Things equal Things be taken away, the Remainders will be equal.*

IV. *If equal Things be added to unequal Things, the Wholes will be unequal.*

V. *If equal Things be taken from unequal Things, the Remainders will be unequal.*

VI. *Things which are double to one and the same Thing, are equal between themselves.*

VII. *Things which are half one and the same Thing, are equal between themselves.*

VIII. *Things which mutually agree together, are equal to one another.*

IX. *The Whole is greater than its Part.*

X. *Two Right Lines do not contain a Space.*

XI. *All Right Angles are equal between themselves.*

XII. *If a Right Line, falling upon two other Right Lines, makes the inward Angles on the same Side thereof, both together, less than two Right Angles, those two Right Lines, infinitely produc'd, will meet each other on that Side where the Angles are less than Right ones.* *

Note, When there are several Angles at one Point, any one of them is express'd by three Letters, of which that at the Vertex of the Angle is plac'd in the Middle. For Example; in the Figure of *Prop.* XIII. *Lib.* I. the Angle contain'd under the Right Lines AB, BC, is called the Angle ABC; and the Angle contained under the Right Lines AB, BE, is called the Angle ABE.

PRO-

* *vide prop.* 28. 1.

PROPOSITION I.

Problem.

To describe an Equilateral Triangle upon a given finite Right Line.

LET AB be the given finite Right Line, upon which it is required to describe an equilateral Triangle.

About the Centre A, with the Distance A B, describe the Circle B C D *; and about the Centre B, with the same Distance B A, describe the Circle A C E *; and from the Point C, where the two Circles cut each other, draw the Right Lines C A, C B †. * *Post.* 3. † *Post.* 1.

Then because A is the Centre of the Circle D B C, A C shall be equal to A B ‡. And because B is the Centre of the Circle C A E, B C shall be equal to B A: But C A hath been proved to be equal to A B; therefore both C A and C B are each equal to A B. But Things equal to one and the same Thing, are equal between themselves *, and consequently A C is equal to C B; therefore the three Sides C A, A B, B C, are equal between themselves. ‡ *Def.* 15. * *Ax.* 1.

And so, *the Triangle* B A C *is an Equilateral one, and is described upon the given finite Right Line* A B; which was to be done.

PROPOSITION II.

Problem.

At a given Point to put a Right Line equal to a Right Line given.

LET the Point given be A, and the given Right Line B C; it is required to put a Right Line at the Point A, equal to the given Right Line B C.

* *Post.* 1. Draw the Right Line A C from the Point A to C *, † 1 *of this.* upon it describe the Equilateral Triangle D A C †; ‡ *Post.* 2. produce D A and D C directly forwards to E and G ‡; about the Centre C, with the Distance B C, describe * *Post.* 3. the Circle B G H *; and about the Centre D, with the Distance D G, describe the Circle K G L.

Now because the Point C is the Centre of the Circle † *Def.* 15. B G H, B C will be equal to C G †; and because D is the Centre of the Circle K G L, the Whole D L will be equal to the Whole D G, the Parts whereof D A and D C are equal; therefore the Remainders ‡ *Ax.* 3. A L, C G, are also equal ‡. But it has been demonstrated, that B C is equal to C G; wherefore both A L and B C are each of them equal to C G. But Things that are equal to one and the same Thing, are equal * *Ax.* 1. to one another *; and therefore likewise A L is equal to B C.

Whence, *the Right Line* A L *is put at the given Point* A, *equal to the given Right Line* B C; which was to be done.

PROPOSITION III.

PROBLEM.

Two unequal Right Lines being given, to cut off a Part from the greater, equal to the lesser.

LET A B and C be the two unequal Right Lines given, the greater whereof is A B; it is required to cut off a Line from the greater A B equal to the lesser C.

* 2 *of this.* Put * a Right Line A D at the Point A, equal to the Line C; and about the Centre A, with the Di- † *Post.* 3. stance A D, describe a Circle D E F †.

Then because A is the Centre of the Circle D E F, A E is equal to A D; and so both A E and C are each equal to A D; whereof A E is likewise equal ‡ *Ax.* 1. to C ‡.

And so, *there is cut off from* A B *the greater of two given Right Lines* A B *and* C, *a Line* A E *equal to the lesser Line* C; which was to be done.

PRO-

PROPOSITION IV.

THEOREM.

If there are two Triangles that have two Sides of the one equal to two Sides of the other, each to each, and the Angle contained by those equal Sides in one Triangle equal to the Angle contained by the correspondent Sides in the other Triangle; then the Base of one of the Triangles shall be equal to the Base of the other, the whole Triangle equal to the whole Triangle, and the remaining Angles of one equal to the remaining Angles of the other, each to each, which subtend the equal Sides.

LET the two Triangles be ABC, DEF, which have two Sides AB, AC, equal to two Sides DE, DF, each to each, that is, the Side AB equal to the Side DE, and the Side AC to DF; and the Angle BAC equal to the Angle EDF. I say, that the Base BC is equal to the Base EF, the Triangle ABC equal to the Triangle DEF, and the remaining Angles of the one equal to the remaining Angles of the other, each to its correspondent, subtending the equal Sides; *viz.* the Angle ABC equal to the Angle DEF, and the Angle ACB equal to the Angle DFE.

For the Triangle ABC being applied to DEF, so as the Point A may co-incide with D, and the Right Line AB with DE, then the Point B will co-incide with the Point E, because AB is equal to DE. And since AB co-incides with DE, the Right Line AC likewise will co-incide with the Right Line DF, because the Angle BAC is equal to the Angle EDF. Wherefore also C will co-incide with F, because the Right Line AC is equal to the Right Line DF. But the Point B co-incides with E, and therefore the Base BC co-incides with the Base EF. For, if the Point B co-inciding with E, and C with F, the Base BC does not co-incide with the Base EF; then two Right Lines will contain a Space, which is impossible*. * *Ax.* 10.
Therefore, *the Base* BC *co-incides with the Base* EF, *and is equal thereto; and consequently the whole Tri-*

angle ABC *will co-incide with the whole Triangle* DEF, *and will be equal thereto; and the remaining*
† Ax. 8. *Angles will co-incide with the remaining Angles*†, *and will be equal to them,* viz. *the Angle* ABC *equal to the Angle* DEF, *and the Angle* ACB *equal to the Angle* DFE; which was to be demonstrated.

PROPOSITION V.

THEOREM.

The Angles at the Base of an Isosceles *Triangle are equal between themselves: And, if the equal Sides be produced, the Angles under the Base shall be equal between themselves.*

LET ABC be an *Isosceles* Triangle, having the Side AB equal to the Side AC; and let the equal Sides AB, AC, be produced directly forwards to D and E. I say, the Angle ABC is equal to the Angle ACB, and the Angle CBD equal to the Angle BCE.

For assume any Point F in the Line BD, and from
* 3 of this. AE cut off the Line AG equal * to AF, and join FC, GB.

Then, because AF is equal to AG, and AB to AC, the two Right Lines FA, AC, are equal to the two Lines GA, AB, each to each, and contain the com-
† 4 of this. mon Angle FAG; therefore the Base FC is equal † to the Base GB, and the Triangle AFC equal to the Triangle AGB, and the remaining Angles of the one equal to the remaining Angles of the other, each to each, subtending the equal Sides, *viz.* the Angle ACF equal to the Angle ABG; and the Angle AFC, equal to the Angle AGB. And because the Whole AF is equal to the Whole AG, and the Part
‡ Ax. 3. AB equal to the Part AC, the Remainder BF ‡ is equal to the Remainder CG. But FC has been proved to be equal to GB; therefore the two Sides BF, FC, are equal to the two Sides CG, GB, each to each, and the Angle BFC equal to the Angle CGB; but they have a common Base BC. Therefore also the Triangle BFC will be equal to the Triangle CGB*, and the remaining Angles of the one

equal

equal to the remaining Angles of the other, each to each, which subtend the equal Sides. And so the Angle F B C is equal to the Angle G C B; and the Angle B C F equal to the Angle C B G. Therefore, because the whole Angle A B G has been proved equal to the whole Angle A C F, and the Part C B G equal to B C F, the remaining Angle A B C will be * equal to the remaining Angle A C B; but these are the Angles at the Base of the Triangle A C B. It hath likewise been proved, that the Angles F B C, G C B, under the Base, are equal; therefore, *the Angles at the Base of Isosceles Triangles are equal between themselves; and if the equal Right Lines be produced, the Angles under the Base will be also equal between themselves*; which was to be demonstrated.

* Ax. 3.

Coroll. Hence every Equilateral Triangle is also Equiangular.

PROPOSITION VI.

THEOREM.

If two Angles of a Triangle be equal, then the Sides subtending the equal Angles will be equal between themselves.

LET A B C be a Triangle, having the Angle A B C equal to the Angle A C B. I say, the Side A B is likewise equal to the Side A C.

For if A B be not equal to A C, let one of them, as A B, be the greater, from which cut off B D equal to A C †, and join D C. Then, because B D is equal to A C, and B C is common, D B, B C, will be equal to A C, C B, each to each, and the Angle D B C equal to the Angle A C B, from the Hypothesis; therefore the Base D C is equal ‡ to the Base A B, and the Triangle D B C equal to the Triangle A C B, a Part to the Whole, which is absurd; therefore A B is not unequal to A C, and consequently is equal to it.

† 3 of this.
‡ 4 of this.

Therefore, *if two Angles of a Triangle be equal between themselves, the Sides subtending the equal Angles are likewise equal between themselves*; which was to be demonstrated.

Coroll. Hence every Equiangular Triangle is also Equilateral.

PROPOSITION VIII.

THEOREM.

On the same Right Line cannot be constituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the same Side, and having the same Ends which the first Right Lines have.

FOR, if it be possible, let two Right Lines AD, DB, equal to two others AC, CB, each to each, be constituted at different Points C and D, towards the same Part CD, and having the same Ends A and B, which the first Right Lines have, so that CA be equal to AD, having the same End A, which CA hath; and CB equal to DB, having the same End B.

Case 1. The Point D cannot fall in the Line AC; for instance at F: For then (AD that is) AF would not be equal to AC.

Case 2. If it be said that D falls within the Triangle ABC; draw CD, and produce BD, BC, to F, and E. Now, since AD is affirmed to be equal to AC, the Angle ADC is equal to the Angle ACD *; and

†FDC consequently the Angle ACD is greater than† [redacted]: Moreover ECD is greater than ACD, therefore ECD is much greater than FDC. But it is also said, that BD is equal to BC, and so the Angle ECD is

* 5 of this. equal to FDC *; whereas it hath been proved to be much greater, which is absurd: Therefore D doth not fall within the Triangle.

Case 3. Suppose D fell without the Triangle ABC; join CD.

Then, because AC is equal to AD, the Angle

† 5 of this. ACD will be equal † to the Angle [redacted], and consequently the Angle ADC is greater than the Angle BCD; wherefore the Angle BDC will be much greater than the Angle BCD. Again, because CB is equal to DB, the Angle BDC will be equal to the Angle BCD; but it has been proved to be much greater, which is impossible. Therefore, *on the same*

Right Line cannot be constituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the same Side, and having the same Ends which the first Right Lines have; which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal.

LET the two Triangles be ABC, DEF, having two Sides, AB, AC, equal to two Sides DE, DF, each to each, *viz.* AB equal to DE, and AC to DF; and let the Base BC be equal to the Base EF. I say, the Angle BAC is equal to the Angle EDF.

For, if the Triangle ABC be applied to the Triangle DEF, so that the Point B may co-incide with E, and the Right Line BC with EF, then the Point C will co-incide with F, because BC is equal to EF. And so, since BC co-incides with EF, BA and AC will likewise co-incide with ED and DF. For if the Base BC should co-incide with EF, and at the same Time the Sides BA, AC, should not co-incide with the Sides ED, DF, but change their Position, as EG, GF, then there would be constituted on the same Right Line two Right Lines, equal to two other Right Lines, each to each, at several Points, on the same Side, having the same Ends. But this is proved to be otherwise †; therefore it is impossible for the Sides BA, AC, not to co-incide with the Sides ED, DF, if the Base BC co-incides with the Base EF; wherefore they will co-incide, and consequently the Angle BAC will co-incide with the Angle EDF, and will be equal to it. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal*; which was to be demonstrated.

† 7 of this.

PRO-

PROPOSITION IX.

PROBLEM.

To cut a given Right-lin'd Angle into two equal Parts.

LET BAC be a given Right-lin'd Angle, which is required to be cut into two equal Parts.

Aſſume any Point D in the Right Line AB, and cut off AE from the Line AC equal to AD †; Join DE, and thereon make ‡ the Equilateral Triangle DEF, and join AF. I ſay, the Angle BAC is cut into two equal Parts by the Line AF.

† 3 *of this.*
‡ 1 *of this.*

For, becauſe AD is equal to AE, and AF is common, the two Sides DA, AF, are equal to the two Sides AE, AF, and the Baſe DF is equal to the Baſe EF; therefore ‡ the Angle DAF is equal to the Angle EAF. Wherefore, *a given Right-lin'd Angle is cut into two equal Parts*; which was to be done.

‡ 8 *of this.*

PROPOSITION X.

PROBLEM.

To cut a given finite Right Line into two equal Parts.

LET AB be a given finite Right Line, required to be cut into two equal Parts.

Upon it make * an Equilateral Triangle ABC, and biſect † the Angle ACB by the Right Line CD. I ſay, the Right Line AB is biſected in the Point D.

* 1 *of this.*
† 9 *of this.*

For, becauſe AC is equal to CB, and CD is common, the Right Lines AC, CD, are equal to the two Right Lines BC, CD, and the Angle ACD equal to the Angle BCD; therefore ‡ the Baſe AD is equal to the Baſe DB. And ſo, *the Right Line AB is biſected in the Point D*; which was to be done.

‡ 4 *of this.*

PRO-

PROPOSITION XI.

PROBLEM.

To draw a Right Line at Right Angles to a given Right Line, from a given Point in the same.

LET AB be the given Right Line, and C the given Point. It is required to draw a Right Line from the Point C, at Right Angles to A B.

Assume any Point D in A C, and make C E equal * to C D; and upon D E make † the Equilateral * 3 *of this.*
Triangle F D E, and join F C. I say, the Right † 1 *of this.*
Line F C is drawn from the Point C, given in the Right Line A B, at Right Angles to A B.

For, because D C is equal to C E, and F C is common, the two Lines D C, C F, are equal to the two Lines E C, C F; and the Base D F is equal to the Base F E. Therefore * the Angle D C F is equal * 8 *of this.*
to the Angle E C F; and they are adjacent Angles. But when a Right Line, standing upon a Right Line, makes the adjacent Angles equal, each of the equal ‡ 10
Angles is ‡ a Right Angle; and consequently D C F, F C E, are both Right Angles. Therefore, *the Right Line* F C *is drawn from the Point* C *at Right Angles to* A B; which was to be done.

PROPOSITION XII.

PROBLEM.

To draw a Right Line perpendicular, upon a given infinite Right Line, from a Point given out of it.

LET AB be the given infinite Line, and C the Point given out of it. It is requir'd to draw a Right Line perpendicular upon the given Right Line AB, from the Point C given out of it.

Assume any Point D on the other Side of the Right Line A B; and about the Centre C, with the Distance C D, describe * a Circle E D G; bisect † E G in H, * *Post.* 3.
and join C G, C H, C E. I say, there is drawn the † 10 *of this.*

Per-

Perpendicular C H on the given infinite Right Line A B, from the Point C given out of it.

For, because GH is equal to HE, and HC is common, GH and HC are each equal to EH and HC, and the Base CG is equal to the Base CE. Therefore

‡ 8 of this. the Angle CHG is equal ‡ to the Angle CHE; and they are adjacent Angles. But when a Right Line, standing upon another Right Line, makes the Angles equal between themselves, each of the equal Angles

* Def. 10. is a Right one *, and the said standing Right Line is call'd a Perpendicular to that which it stands on. Therefore, CH *is drawn perpendicular, upon a given infinite Right Line, from a given Point out of it*; which was to be done.

PROPOSITION XIII.

THEOREM.

When a Right Line, standing upon a Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles.

FOR let a Right Line AB, standing upon the Right Line C D, make the Angles C B A, A B D. I say, the Angles C B A, A B D, are either two Right Angles, or both together equal to two Right Angles.

* Def. 10. For if C B A be equal to A B D, they are * each of

† 11 of this. them Right Angles: But if not, draw † B E from the Point B, at Right Angles to C D. Therefore the Angles C B E, E B D, are two Right Angles: And because CBE is equal to both the Angles CBA, ABE,

add add the Angle EBD, which is common; and the two

‡ Ax. 2. Angles C B E, E B D, together, are ‡ equal to the three Angles C B A, A B E, E B D, together. Again, because the Angle D B A is equal to the two Angles D B E, E B A, together, add the common Angle A B C, and the two Angles D B A, A B C, are equal to the three Angles D B E, E B A, A B C, together. But it has been proved, that the two Angles C B E, EBD, together, are likewise equal to these three Angles: But Things that are equal to one and the same,

* Ax. 1. are * equal between themselves. Therefore likewise the Angles C B E, E B D, together, are equal to the

Angles

Angles D B A, A B C, together; but C B E, E B D, are two Right Angles. Therefore the Angles D B A, A B C, are both together equal to two Right Angles. Wherefore, *when a Right Line, standing upon another Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles*; which was to be demonstrated.

PROPOSITION XIV.

THEOREM.

If to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one strait Line.

FOR let two Right Lines B C, B D, drawn from contrary Parts to the Point B, in any Right Line A B, make the adjacent Angles A B C, A B D, both together equal to two Right Angles. I say, B C, B D, make but one Right Line.

For if B D, C B, do not make one strait Line, let C B and B E make one.

Then, because the Right Line AB stands upon the Right Line CBE, the Angles ABC, ABE, together, will be equal * to two Right Angles. But the Angles A B C, A B D, together, are also equal to two Right Angles. Now taking away the common Angle ABC, the remaining Angle A B E is equal to the remaining Angle A B D, the less to the greater, which is impossible. Therefore B E, B C, are not one strait Line. And in the same manner it is demonstrated, that no other Line but B D is in a strait Line with CB; wherefore CB, BD, shall be in one strait Line. Therefore, *if to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one strait Line*; which was to be demonstrated.

* 13 of this.

PROPOSITION XV.

THEOREM.

If two Right Lines mutually cut each other, the opposite Angles are equal.

LET the two Right Lines AB, CD, mutually cut each other in the Point E. I say, the Angle AEC is equal to the Angle DEB; and the Angle CEB equal to the Angle AED.

For, because the Right Line AE, standing on the Right Line CD, makes the Angles CEA, AED; * 13 *of this.* these both together shall be equal * to two Right Angles. Again, because the Right Line DE, standing upon the Right Line AB, makes the Angles AED, DEB; these Angles together are * equal to two Right Angles. But it has been proved, that the Angles CEA, AED, are likewise together equal to two Right Angles. Therefore the Angles CEA, AED, are equal to the Angles AED, DEB. Take away the common † *Ax.* 3. Angle AED, and the Angle remaining CEA is † equal to the Angle remaining BED. For the same Reason, the Angle CEB shall be equal to the Angle DEA. Therefore, *if two Right Lines mutually cut each other, the opposite Angles are equal*; which was to be demonstrated.

Coroll. 1. From hence it is manifest, that two Right Lines, mutually cutting each other, make Angles at the Section equal to four Right Angles.

Coroll. 2. All the Angles, constituted about the same Point, are equal to four Right Angles.

PRO-

PROPOSITION XVI.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is greater than either of the inward opposite Angles.

LET ABC be a Triangle, and one of its Sides BC be produced to D. I say, the outward Angle ACD is greater than either of the inward Angles CBA, or BAC.

For bisect AC in E*, and join BE, which produce to F, and make EF equal to BE †. Moreover, join FC, and produce AC to G. * 10 of this. † 3 of this.

Then, because AE is equal to EC, and BE to EF, the two Sides AE, EB, are equal to the two Sides CE, EF, each to each, and the Angle AEB † equal to the Angle FEC; for they are opposite Angles. Therefore the Base AB is ‡ equal to the Base FC; and the Triangle AEB equal to the Triangle FEC; and the remaining Angles of the one equal to the remaining Angles of the other, each to each, subtending the equal Sides. Wherefore the Angle BAE is equal to the Angle ECF; but the Angle ECD is greater than the Angle ECF; therefore the Angle ACD is greater than the Angle BAE. After the same manner, if the Right Line BC be bisected, we demonstrate that the Angle BCG, and consequently its equal, the Angle ACD *, is greater than the Angle ABC. Therefore, *one Side of any Triangle being produced, the outward Angle is greater than either of the inward opposite Angles*; which was to be demonstrated. † 15 of this. ‡ 4 of this. * 15 of this.

PROPOSITION XVII.

THEOREM.

Two Angles of any Triangle together, howsoever taken, are less than two Right Angles.

LET ABC be a Triangle. I say, two Angles of it together, howsoever taken, are less than two Right Angles.

For produce BC to D.

Then, because the outward Angle ACD of the Triangle ABC is greater * than the inward opposite Angle ABC; if the common Angle ACB be added, the Angles ACD, ACB, together, will be greater than the Angles ABC, ACB, together: But ACD, ACB, are † equal to two Right Angles. Therefore ABC, BCA, are less than two Right Angles. In the same manner we demonstrate, that the Angles BAC, ACB, as also CAB, ABC, are less than two Right Angles. Therefore, *two Angles of any Triangle together, howsoever taken, are less than two Right Angles*; which was to be demonstrated.

* 16 *of this.*
† 13 *of this.*

PROPOSITION XVIII.

THEOREM.

The greater Side of every Triangle subtends the greater Angle.

LET ABC be a Triangle, having the Side AC greater than the Side AB. I say, the Angle ABC is greater than the Angle BCA.

For, because AC is greater than AB, AD may be made equal to AB ‡, and BD be joined.

Then, because ADB is an outward Angle of the Triangle BDC, it will be * greater than the inward opposite Angle DCB. But ADB is † equal to ABD; because the Side AB is equal to the Side AD. Therefore the Angle ABD is likewise greater than the Angle ACB; and consequently ABC shall be much greater than ACB. Wherefore, *the greater Side of every Triangle subtends the greater Angle*; which was to be demonstrated.

‡ 3 *of this.*
* 16 *of this.*
† 5 *of this.*

PRO-

PROPOSITION XIX.

THEOREM.

The greater Angle of every Triangle subtends the greater Side.

LET ABC be a Triangle, having the Angle ABC greater than the Angle BCA. I say, the Side AC is greater than the Side AB.

For, if it be not greater, AC is either equal to AB, or less than it. It is not equal to it, because then the Angle ABC would be equal * to the Angle ACB; but it is not: Therefore AC is not equal to AB. Neither will it be less; for then the Angle ABC would be † less than the Angle ACB; but it is not. Therefore AC is not less than AB. But likewise it has been proved not to be equal to it: Wherefore AC is greater than AB. Therefore, *the greater Angle of every Triangle subtends the greater Side*; which was to be demonstrated.

* 5 *of this.*

† 18 *of this.*

PROPOSITION XX.

THEOREM.

Two Sides of any Triangle, howsoever taken, are together greater than the Third Side.

LET ABC be a Triangle: I say, two Sides thereof, howsoever taken, are together greater than the third Side; *viz.* the Sides BA, AC, are greater than the Side BC; and the Sides AB, BC, greater than the Side AC; and the Sides BC, CA, greater than the Side AB.

For produce BA to the Point D, so that AD be equal * to AC; and join DC.

Then, because DA is equal to AC, the Angle ADC shall be equal † to the Angle ACD. But the Angle BCD is greater than the Angle ACD. Wherefore the Angle BCD is greater than the Angle ADC; and because DCB is a Triangle, having the Angle BCD greater than the Angle BDC, and the greater Angle

* 3 *of this.*

† 5 *of this.*

‡ 19 *of this.* Angle ſubtends ‡ the greater Side; the Side D B will be greater than the Side B C. But D B is equal to B A and A C together. Wherefore the Sides B A, A C, together, are greater than the Side B C. In the ſame manner we demonſtrate, that the Sides A B, B C, together, are greater than the Side C A; and the Sides BC, CA, together, are greater than the Side A B. Therefore, *two Sides of any Triangle, howſoever taken, are together greater than the third Side*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

If two Right Lines be drawn from the extreme Points of one Side of a Triangle, to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle.

FOR let two Right Lines BD, DC, be drawn from the Extremes B, C, of the Side B C of the Triangle ABC, to the Point D within the ſame. I ſay, BD, DC, are leſs than BA, AC, the other two Sides of the Triangle, but contain an Angle B D C greater than the Angle B A C.

For produce B D to E.

Then, becauſe two Sides of every Triangle together
* 20 *of this.* are * greater than the third, B A, A E the two Sides of the Triangle A B E, are greater than the Side B E. Now, add E C, which is common, and the Sides
† *Ax* 4. B A, A C, will be † greater than B E, E C.

Again, becauſe CE, ED, the two Sides of the Triangle C E D, are greater than the Side C D, add D B, which is common, and the Sides C E, E B, will be greater than C D, D B. But it has been proved, that BA, AC, are greater than BE, EC. Wherefore BA, AC, are much greater than BD, DC. Again, becauſe
‡ 16 *of this.* the outward Angle of every Triangle ‡ is greater than the inward and oppoſite one; B D C, the outward Angle of the Triangle CDE, ſhall be greater than the Angle C E D. For the ſame Reaſon, C E D, the outward Angle of the Triangle A B E, is likewiſe greater than

than the Angle B A C; but the Angle B D C has been proved to be greater than the Angle C E D. Wherefore the Angle B D C ſhall be much greater than the Angle B A C. And ſo, *if two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle*; which was to be demonſtrated.

PRPOSITION XXII.

PROBLEM.

To deſcribe a Triangle of three Right Lines, which are equal to three others given: But it is requiſite, that any two of the Right Lines taken together be greater than the third; becauſe two Sides of a Triangle, howſoever taken, are together greater than the third Side.

LET A, B, C, be three Right Lines given, two of which, any ways taken, are greater than the third; *viz.* A and B together greater than C; A and C greater than B; and B and C greater than A. Now it is required to make a Triangle of three Right Lines equal to A, B, C: Let there be one Right Line D E, terminated at D, but infinite towards E; and take ‡ D F equal to A, F G equal to B, and G H equal to C; and about the Centre F, with the Diſtance F D, deſcribe a Circle DKL †; and about the Centre G, with the Diſtance G H, deſcribe another Circle K L H, and join K F, K G. I ſay, the Triangle K F G is made of three Right Lines, equal to A, B, C; for, becauſe the Point F is the Centre of the Circle D K L, F K ſhall be equal to F D ‡: But F D is equal to A; therefore F K is alſo equal to A. Again, becauſe the Point G is the Centre of the Circle L K H, G K will be ‡ equal to G H: But G H is equal to C; therefore ſhall G K be alſo equal to C: But F G is likewiſe equal to B; and conſequently the three Right Lines K F, F G, K G, are equal to the three Right Lines A, B, C. Wherefore, *the Triangle* K F G *is made of three Right Lines* K F, F G, K G, *equal to the three given Lines* A, B, C, which was to be done.

‡ 3 *of this.*

† 3 *Poſt.*

‡ *Def.* 15.

PROPOSITION XXIII.

PROBLEM.

With a given Right Line, and at a given Point in it, to make a Right-lin'd Angle equal to a Right-lin'd Angle given.

LET the given Right Line be AB, and the Point given therein A, and the given Right-lin'd Angle DCE. It is required to make a Right-lin'd Angle at the given Point A, with the given Right Line AB, equal to the given Right-lin'd Angle DCE.

Assume the Points D and E at Pleasure in the Lines CD, CE, and draw DE; then, of three Right Lines ‡ 22 of this. equal to CD, DE, EC, make ‡ a Triangle AFG, so that AF be equal to CD, AG to CE, and FG to DE.

Then, because the two Sides DC, CE, are equal to the two Sides FA, AG, each to each, and the Base DE equal to the Base FG; the Angle DCE shall be † 8 of this. † equal to the Angle FAG. Therefore, *the Right-lin'd Angle* FAG *is made, at the given Point* A, *in the given Right Line* AB, *equal to the given Right-lin'd Angle* DCE; which was to be done.

PROPOSITION XXIV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Angle of the one contained under the equal Right Lines, greater than the correspondent Angle of the other; then the Base of the one will be greater than the Base of the other.

LET there be two Triangles ABC, DEF, having two Sides AB, AC, equal to the two Sides DE, DF, each to each, *viz.* the Side AB equal to the Side DE, and the Side AC equal to DF; and let the Angle BAC be greater than the Angle EDF. I say, the Base BC is greater than the Base EF.

For

For becauſe the Angle B A C is greater than the Angle E D F; make an ‡ Angle E D G at the Point D in the Right Line D E, equal to the Angle B A C; and make † D G equal to either A C or D F, and join E G, F G. ‡ 23 *of this.* † 3 *of this.*

Caſe 1. When E G falls above E F; then, becauſe AB is equal to D E, and A C to D G, the two Sides B A, A C, are each equal to the two Sides E D, D G, and the Angle B A C equal to the Angle E D G: Therefore the Baſe B C is equal ‡ to the Baſe E G. Again, becauſe DG is equal to DF, the Angle DFG is † equal to the Angle D G F; and ſo the Angle D F G is greater than the Angle E G F: And conſequently the Angle E F G is much greater than the Angle E G F. And becauſe E F G is a Triangle, having the Angle E F G greater than the Angle E G F; and the greateſt Angle ſubtends ‖ the greateſt Side, the Side E G ſhall be greater than the Side E F. But the Side E G is equal to the Side B C: Whence B C is likewiſe greater than E F. ‡ 4 *of this.* † 5 *of this.* ‖ 19 *of this.*

Caſe 2. When E G falls upon E F; then E G is greater than E F: And conſequently B C is greater than E F.

Caſe 3. When E G falls below E F; then D G, E G, are ‡ together greater than D F and E F together, and by taking away the equals D G, D F, there remains E G greater than E F †. Therefore B C, which is equal to E G, will be alſo greater than E F. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Angle of the one contained under the equal Right Lines, greater than the correſpondent Angle of the other; then the Baſe of the one will be greater than the Baſe of the other;* which was to be demonſtrated. ‡ 21 *of this.* † *Ax.* 5.

PROPOSITION XXV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles contained by the equal Sides, the one greater than the other. *

LET there be two Triangles ABC, DEF, having two Sides A B, A C, each equal to two Sides D E, D F, *viz.* the Side A B equal to the Side D E, and the Side A C to the Side D F; but the Base B C greater than the Base E F; I say, the Angle B A C is also greater than the Angle E D F.

For if it be not greater, it will be either equal or less. But the Angle B A C is not equal to the Angle E D F; for if it was, the Base B C would be ‡ equal to the Base E F; but it is not: Therefore the Angle B A C is not equal to the Angle E D F, neither will it be lesser; for if it should, the Base B C would be † less than the Base E F; but it is not. Therefore the Angle B A C is not less than the Angle E D F; but it has likewise been proved not to be equal to it. Wherefore the Angle B A C is necessarily greater than the Angle E D F. *If*, therefore, *two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles contained by the equal Sides, the one greater than the other*; which was to be demonstrated.

‡ 4 of this.

† 24 of this.

*The reverse of the foregoing.

PRO-

PROPOSITION XXVI.

THEOREM.

If two Triangles have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to his correspondent Side; and the remaining Angle of the one equal to the remaining Angle of the other. *

LET there be two Triangles A B C, D E F, having two Angles A B C, B C A, of the one, equal to two Angles D E F, E F D, of the other, each to each, that is, the Angle A B C equal to the Angle DEF, and the Angle BCA equal to the Angle EFD. And let one Side of the one be equal to one Side of the other, which first let be the Side lying between the equal Angles, *viz.* the Side BC equal to the Side E F. I say, the remaining Sides of the one Triangle will be equal to the remaining Sides of the other, each to each, that is, the Side A B equal to the Side D E, and the Side A C equal to the Side D F, and the remaining Angle B A C equal to the remaining Angle E D F.

For if the Side A B be not equal to the Side D E, one of them will be the greater, which let be A B, make G B equal to D E, and join G C.

Then, because BG is equal to DE, and BC to EF, the two Sides G B, B C, are equal to the two Sides D E, E F, each to each; and the Angle G B C equal to the Angle D E F. The Base G C is ‡ equal to the Base D F, and the Triangle G B C to the Triangle D E F, and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Therefore the Angle G C B is equal to the Angle D F E. But the Angle D F E, by the Hypothesis, ‡ 4 *of this.*

*The reverse of prop. 4.

thesis, is equal to the Angle BCA; and so the Angle BCG is likewise equal to the Angle BCA, the less to the greater, which cannot be. Therefore AB is not unequal to DE, and consequently is equal to it. And so the two Sides AB, BC, are equal to the two Sides DE, EF, and the Angle ABC equal to the Angle

‡ 4 of this. DEF: And consequently the Base AC ‡ is equal to the Base DF, and the remaining Angle BAC equal to the remaining Angle EDF.

Secondly, Let the Sides that are subtended by the equal Angles be equal, as AB equal to DE. I say, the remaining Sides of the one Triangle are equal to the remaining Sides of the other, *viz.* AC to DF, and BC to EF; and also the remaining Angle BAC, to the remaining Angle EDF.

For if BC be unequal to EF, one of them is the greater, which let be BC, if possible, and make BH equal to EF, and join AH.

Now, because BH is equal to EF, and AB to DE, the two Sides AB, BH, are equal to the two Sides DE, EF, each to each, and they contain equal Angles:

‡ 4. Therefore the Base AH is ‡ equal to the Base DF; and the Triangle ABH shall be equal to the Triangle DEF, and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides; and so the Angle BHA is equal to the

† From the Hyp. Angle EFD. But EFD is † equal to the Angle BCA; and consequently the Angle BHA is equal to the Angle BCA: Therefore the outward Angle BHA of the Triangle AHC, is equal to the inward

‡ 16 of this. and opposite Angle BCA; which is ‡ impossible: Whence BC is not unequal to EF; therefore it is equal to it. But AB is also equal to DE. Wherefore the two Sides AB, BC, are equal to the two Sides DE, EF, each to each; and they contain equal Angles. And so the Base AC is equal to the Base DF, the Triangle BAC to the Triangle DEF, and the remaining Angle BAC equal to the remaining Angle EDF. *If, therefore, two Triangles have two Angles equal, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to its correspon-*

dent

dent Side, and the remaining Angle of the one equal to the remaining Angle of the other; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel.

LET the Right Line E F, falling upon two Right Lines AB, CD, make the alternate Angles AEF, E F D, equal between themselves. I say, the Right Line A B is parallel to C D.

For if it be not parallel, A B and C D, produced towards B and D, or towards A and C, will meet: Now let them be produced towards B and D, and meet in the Point G.

Then the outward Angle A E F of the Triangle G E F is ‡ greater than the inward and opposite Angle E F G, and also equal † to it; which is absurd. Therefore A B and C D, produced towards B and D, will not meet each other. By the same Way of Reasoning, neither will they meet, being produced towards C and A. But Lines that meet each other on neither Side, are ‡ parallel between themselves. Therefore A B is parallel to C D. Therefore, *if a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel*; which was to be demonstrated.

‡ *16 of this.*
† *From the Hyp.*
‡ *Def.* 35.

PRO-

PROPOSITION XXVIII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the outward Angle with the one Line equal to the inward and opposite Angle with the other on the same Side; or the inward Angles on the same Side together equal to two Right Angles, the two Right Lines shall be parallel between themselves.

LET the Right Line EF falling upon two Right Lines AB, CD, make the outward Angle EGB equal to the inward and opposite Angle GHD; or the inward Angles BGH, GHD, on the same Side together equal to two Right Angles. I say, the Right Line AB is parallel to the Right Line CD.

‡ *From the Hyp.* † *15 of this.* For, because the Angle EGB is ‡ equal to the Angle GHD, and the Angle EGB † equal to the Angle AGH, the Angle AGH shall be equal to the Angle GHD; but these are alternate Angles. ‡ *27 of this.* Therefore AB is ‡ parallel to CD.

Again, because the Angles BGH, GHD, are equal † *13 of this.* to two Right Angles, and AGH, BGH, are † equal to two Right ones, the Angles AGH, BGH, will be equal to the Angles BGH, GHD; and if the common Angle BGH be taken from both, there will remain the Angle AGH equal to the Angle GHD; but these are alternate Angles. Therefore AB is parallel to CD. *If, therefore, a Right Line, falling upon two Right Lines, makes the outward Angle with the one Line equal to the inward and opposite Angle with the other on the same Side, or the inward Angles on the same Side together equal to two Right Angles, the two Right Lines shall be parallel between themselves*; which was to be demonstrated.

Coroll. Hence ax. 12.

PRO-

PROPOSITION XXIX.

THEOREM.

If a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side; and the inward Angles on the same Side together equal to two Right Angles. *

LET the Right Line E F fall upon the parallel Right Lines A B, C D. I say, the alternate Angles A G H, G H D, are equal between themselves; the outward Angle E G B is equal to the inward one G H D, on the same Side; and the two inward ones, B G H, G H D, on the same Side, are together equal to two Right Angles.

For if AGH be unequal to GHD, one of them will be the greater. Let this be A G H; then because the Angle AGH is greater than the Angle GHD, add the common Angle B G H to both: And so the Angles A G H, B G H, together, are greater than the Angles BGH, GHD, together. But the Angles AGH, BGH, are equal to two Right ones ‡. Therefore BGH, GHD, are less than two Right Angles. And so the Lines A B, C D, infinitely produced † will meet each other; but because they are parallel they will not meet. Therefore the Angle A G H is not unequal to the Angle G H D. Wherefore it is necessarily equal to it.

‡ 13 *of this.*

† *Ax.* 12.

But the Angle AGH is ‡ equal to the Angle EGB: Therefore E G B is also equal to G H D.

‡ 15 *of this.*

Now add the common Angle B G H; and then EGB, BGH, together, are equal to BGH, GHD, together; but EGB, and BGH, are equal to two Right Angles. Therefore also B G H, and G H D shall be equal to two Right Angles. Wherefore, *if a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side; and the inward Angles on the same Side together equal to two Right Angles;* which was to be demonstrated.

PRO-

PROPOSITION XXX.

THEOREM.

Right Lines, parallel to one and the same Right Line, are also parallel between themselves.

LET AB and CD be Right Lines, each of which is parallel to the Right Line EF. I say, AB is also parallel to CD. For let the Right Line GK fall upon them.

‡ 29 *of this.* Then, because the Right Line GK falls upon the parallel Right Lines AB, EF, the Angle AGH is ‡ equal to the Angle GHF; and because the Right Line GK falls upon the parallel Right Lines EF, CD, the Angle GHF is equal to the Angle GKD †. But it has been proved that the Angle AGK is also equal to the Angle GHF. Therefore AGK is equal to GKD, and they are alternate Angles; whence AB is parallel
† 27 *of this.* to CD †. And so, *Right Lines, parallel to one and the same Right Line, are parallel between themselves*; which was to be demonstrated.

PROPOSITION XXXI.

PROBLEM.

To draw a Right Line through a given Point parallel to a given Right Line.

LET A be a Point given, and BC a Right Line given. It is required to draw a Right Line thro' the Point A, parallel to the Right Line BC.

Assume any Point D in BC, and join AD; then
‡ 23 *of this.* make ‡ an Angle DAE, at the Point A, with the Line DA, equal to the Angle ADC, and produce EA strait forwards to F.

Then, because the Right Line AD, falling on two Right Lines BC, EF, makes the alternate Angles EAD, ADC, equal between themselves, EF shall be
† 27 *of this.* † parallel to BC. Therefore, *the Right Line* EAF *is drawn thro' the given Point* A, *parallel to the given Right Line* BC; which was to be done.

PROPOSITION XXXII.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is equal to both the inward and opposite Angles; and the three inward Angles of a Triangle are equal to two Right Angles. †

LET ABC be a Triangle, one of whose Sides BC is produced to D. I say, the outward Angle ACD is equal to the two inward and opposite Angles CAB, ABC; and the three inward Angles of the Triangle, *viz.* ABC, BCA, CAB, are equal to two Right Angles.

For let CE be drawn ‡ thro' the Point C, parallel to the Right Line AB. Then, because AB is parallel to CE, and AC falls upon them, the alternate Angles BAC, ACE are † equal between themselves. Again, because AB is parallel to CE, and the Right Line BD falls upon them, the outward Angle ECD is † equal to the inward and opposite one ABC; but it has been proved, that the Angle ACE is equal to the Angle BAC. Wherefore the whole outward Angle ACD is equal to both the inward and opposite Angles BAC, ABC. And if the Angle ACB, which is common, be added, the two Angles ACD, ACB, are equal to the three Angles ABC, BAC, ACB; but the Angles ACD, ACB, are ‡ equal to two Right Angles. Therefore also shall the Angles ACB, CBA, CAB, be equal to two Right Angles. Wherefore, *if one Side of any Triangle be produced, the outward Angle is equal to both the inward and opposite Angles, and the three inward Angles of a Triangle are equal to two Right Angles*; which was to be demonstrated.

‡ 31 *of this.*

† 29 *of this.*

‡ 13 *of this.*

Coroll. 1. All the three Angles of any one Triangle, taken together, are equal to all the three Angles of any other Triangle, taken together.

Coroll. 2. If two Angles of any one Triangle, either separately, or taken together, be equal to two Angles of any other Triangle; then the remaining Angle of

† vide prop. 16.

of the one Triangle will be equal to the remaining Angle of the other.

Coroll. 3. If one Angle of a Triangle be a Right Angle, the other two Angles together make one Right Angle.

Coroll. 4. If the Angle included between the equal Legs of an Isosceles Triangle be a Right one, each of the other Angles at the Base will be half a Right Angle.

Coroll. 5. Any Angle in an Equilateral Triangle is equal to one Third of two Right Angles, or two Thirds of one Right Angle.

Coroll. 6. Hence it appears, that if one Angle of any Triangle be equal to the other two, that is a Right one; because that the Angle adjacent to this Right one, is equal to the other two. But when adjacent Angles are equal, they are necessarily Right ones.

THEOREM I.

All the inward Angles of any Right-lin'd Figure whatsoever, make twice as many Right Angles, abating four, as the Figure has Sides.

FOR any Right-lin'd Figure may be resolved into as many Triangles, abating two, as it hath Sides. For Example, if a Figure has four Sides, it may be resolved into two Triangles: If a Figure hath five Sides, it may be resolved in three Triangles; if six, into four; and so on. Wherefore (*by* Prop. XXXII.) *the Angles of all these Triangles are equal to twice as many Right Angles as there are Triangles: But the Angles of all the Triangles are equal to the inward Angles of the Figure. Therefore all the inward Angles of the Figure are equal to twice as many Right Angles as there are Triangles, that is, twice as many Right Angles, taking away four, as the Figure hath Sides.* W. W. D.

THE-

THEOREM II.

All the outward Angles of any Right-lined Figure, together, make four Right Angles.

FOR the outward Angles, together with the inward ones, make twice as many Right Angles as the Figure has Sides; But from the last Theorem, all the inward Angles, together, make twice as many Right Angles, abating four, as the Figure hath Sides. Wherefore the outward Angles are, all together, equal to four Right Angles. W. W. D.

PROPOSITION XXXIII.

THEOREM.

Two Right Lines, which join two equal and parallel Right Lines, towards the same Parts, are also equal and parallel.

LET the parallel and equal Right Lines AB, CD, be joined, towards the same Parts, by the Right Lines AC, BD. I say, AC, BD, are equal and parallel.

For draw B C.

Then, because AB is parallel to CD, and BC falls upon them, the alternate Angles A B C, B C D, are * equal. Again, because A B is equal to C D, and B C is common; the two Sides A B, B C, are equal to the two Sides B C, C D; But the Angle A B C is also equal to the Angle BCD; therefore the Base AC is † equal to the Base B D: And the Triangle A B C equal to the Triangle B C D; and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Wherefore the Angle A C B is equal to the Angle C B D. And because the Right Line B C, falling upon two Right Lines A C, B D, makes the alternate Angles A C B, [illegible] equal between themselves; A C is ‡ parallel to B D. But it has been proved also to be equal to it. Therefore, *two Right Lines, which join two equal and parallel Right Lines, towards the same Parts, are also equal and parallel*; which was to be demonstrated.

* 29 of this.
† 4 of this.
CBD
‡ 29 of this.

Defin.

Defin. *A Parallelogram is a Quadrilateral Figure, each of whose opposite Sides are parallel.*

PROPOSITION XXXIV.

THEOREM.

The opposite Sides and opposite Angles of any Parallelogram are equal; and the Diameter divides the same into two equal Parts.

LET ABCD be a Parallelogram, whose Diameter is BC. I say, the opposite Sides and opposite Angles are equal between themselves, and the Diameter BC bisects the Parallelogram.

For, because AB is parallel to CD, and the Right Line BC falls on them, the alternate Angles ABC,
* 29 of this. BCD, are * equal between themselves; again, because AC is parallel to BD, and BC falls upon them, the alternate Angles ACB, and CBD, are equal to one another. Wherefore ABC, CBD, are two Triangles, having two Angles ABC, BCA, of the one, equal to two Angles BCD, CBD, of the other, each to each; and likewise one Side of the one equal to one Side of the other, *viz.* the Side BC between the equal Angles, which is common. Therefore the remaining
† 26 of this. Sides shall be † equal to the remaining Sides, each to each, and the remaining Angle to the remaining Angle. And so the Side [illegible] is equal to the Side CD, the Side AC to BD, and the Angle BAC to the Angle BDC. And because the Angle ABC is equal to the Angle BCD, and the Angle CBD to the Angle ACB; therefore the whole Angle ABD is equal to the whole Angle ACD: But it has been proved, that the Angle BAC is also equal to the Angle BDC.

Wherefore, *the opposite Sides and Angles of any Parallelogram* [illegible]*ual between themselves.*

I say, moreover, that the Diameter bisects it. For because AB is equal to CD, and BC is common, the two Sides AB, BC, are each equal to the two Sides DC, CB; and the Angle ABC is also equal to the Angle DCB. Therefore the Base AC is
‡ 4 of this. ‡ equal to the Base DB[illegible]nd the Triangle ABC is equal

equal to the Triangle BCD. Wherefore, the *Diameter* BC *bisects the Parallelogram* ACDB; which was to be demonstrated.

PROPOSITION XXXV.

THEOREM.

Parallelograms constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET ABCD, EBCF, be Parallelograms constituted upon the same Base BC, and between the same Parallels AF and BC. I say, the Parallelogram ABCD is equal to the Parallelogram EBCF.

For, because ABCD is a Parallelogram, AD is * equal to BC; and for the same Reason EF is equal to BC; wherefore AD shall be † equal to EF; but DE is common, Therefore the whole AE is ‡ equal to the whole DF. But AB is equal to DC; wherefore EA, AB, the two Sides of the Triangle ABE, are equal to the two Sides FD, DC, each to each; and the Angle FDC * equal to the Angle EAB, the outward one to the inward one. Therefore the Base EB is † equal to the Base CF, and the Triangle EAB to the Triangle FDC. If the common Triangle DGE be taken from both, there will remain ‡ the Trapezium ABGD, equal to the Trapezium FCGE; and if the Triangle GBC, which is common, be added, the Paralellogram ABCD will be equal to the Parallelogram EBCF, Therefore, *Parallelograms constituted upon the same Base, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* 34 *of this.*
† *Ax.* 1.
‡ *Ax.* 2.
* 29 *of this.*
† 4 *of this.*
‡ *Ax.* 3.

PROPOSITION XXXVI.

THEOREM.

Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Parallelograms A B C D, E F G H, be constituted upon the equal Bases B C, F G, and between the same Parallels A H, B G. I say, the Parallelogram A B C D is equal to the Parallelogram E F G H.

* Hyp. For join B E, C H. Then because B C is * equal to F G, and F G to E H; B C will be likewise equal to EH; and they are parallel, and BE, CH, join them. But two Right Lines joining Right Lines, which are

† 33 of this. equal and parallel, towards the same Parts, are † equal and parallel: Wherefore E B C H is a Parallelogram,

‡ 35 of this. and is ‡ equal to the Parallelogram A B C D; for it has the same Base B C, and is constituted between the same Parallels B C, A H. For the same Reason, the Parallelogram E F G H is equal to the same Parallelogram E B C H. Therefore the Parallelogram A B C D shall be equal to the Parallelogram E F G H. And so *Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET the Triangles A B C, D B C, be constituted upon the same Base B C, and between the same Parallels A D, B C. I say, the Triangle ABC is equal to the Triangle D B C.

For produce A D both Ways to the Points E and

* 31 of this. F; and through B draw * B E parallel to C A; and through C, C F, parallel to B D.

Wherefore both E B C A, D B C F, are Parallelograms; and the Parallelogram E B C A is * equal to the Parallelogram D B C F; for they stand upon the same Base B C, and between the same Parallels B C, E F. But the Triangle A B C is † one half of the Parallelogram EBCA, because the Diameter AB bisects it; and the Triangle DBC is one half of the Parallelogram DBCF, for the Diameter DC bisects it. But Things that are the Halves of equal Things, are ‡ equal between themselves. Therefore the Triangle A B C is equal to the Triangle D B C. Wherefore, *Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* 35 of this.
† 34 of this.
‡ Ax. 7.

PROPOSITION XXXVIII.

THEOREM.

Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Triangles A B C, D C E, be constituted upon the equal Bases B C, C E, and between the same Parallels B E, A D. I say, the Triangle A B C is equal to the Triangle D C E.

For, produce AD both Ways to the Points, G, H; through B draw * B G parallel to C A; and through E, E H, parallel to D C.

Wherefore both G B C A, D C E H, are Parallelograms; and the Parallelogram G B C A is † equal to the Parallelogram D C E H: For they stand upon equal Bases, B C, C E, and between the same Parallels B E, G H. But the Triangle A B C is ‡ one half of the Parallelogram G B C A, for the Diameter A B bisects it; and the Triangle D C E ‡ is one half of the Parallelogram D C E H, for the Diameter D E bisects it. But Things that are the Halves of equal Things, are * equal between themselves. Therefore the Triangle A B C is equal to the Triangle D C E. Wherefore, *Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* 31 of this.
† 36 of this.
‡ 34 of this.
* Ax. 7.

PROPOSITION XXXIX.

THEOREM.

Equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels.

LET ABC, DBC, be equal Triangles, constituted upon the same Base BC, on the same Side. I say they are between the same Parallels. For, let AD be drawn. I say, AD is parallel to BC.

* 31 *of this.* For, if it be not parallel, draw * the Right Line AE thro' the Point A, parallel to BC, and draw EC.
† 37 *of this.* Then the Triangle ABC † is equal to the Triangle EBC; for it is upon the same Base BC, and between the same Parallels BC, AE. But the Triangle ABC
‡ *From Hyp.* is ‡ equal to the Triangle DBC. Therefore the Triangle DBC is also equal to the Triangle EBC, a greater to a less, which is impossible. Wherefore AE is not parallel to BC: And by the same Way of Reasoning we prove, that no other Line but AD is parallel to BC. Therefore AD is parallel to BC. Wherefore, *equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels*; which was to be demonstrated.

PROPOSITION XL.

THEOREM.

Equal Triangles constituted upon equal Bases, on the same Side, are between the same Parallels.

LET ABC, CDE, be equal Triangles, constituted upon equal Bases BC, CE. I say, they are between the same Parallels. For, let AD be drawn. I say, AD is parallel to BE.

* 31 *of this.* For, if it be not, let AF be drawn * through A, parallel to BE, and draw FE.
† 38 *of this.* Then the Triangle ABC is † equal to the Triangle FCE; for they are constituted upon equal Bases, and between the same Parallels BE, AF. But the Triangle ABC is equal to the Triangle DCE. There-

fore

fore the Triangle DCE ſhall be equal to the Triangle FCE, the greater to the leſs, which is impoſſible. Wherefore AF is not parallel to BE. And in this manner we demonſtrate, that no Right Line can be parallel to BE, but AD. Therefore AD is parallel to BE. And ſo, *equal Triangles conſtituted upon equal Baſes, on the ſame Side, are between the ſame Parallels*; which was to be demonſtrated.

PROPOSITION XLI.

THEOREM.

If a Parallelogram and a Triangle have the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double to the Triangle.

LET the Parallelogram ABCD, and the Triangle EBC, have the ſame Baſe, and be between the ſame Parallels, BC, AE. I ſay, the Parallelogram ABCD is double the Triangle EBC.

For join AC.

Now the Triangle ABC is * equal to the Triangle EBC; for they are both conſtituted upon the ſame Baſe BC, and between the ſame Parallels BC, AE. But the Parallelogram ABCD is † double the Triangle ABC, ſince the Diameter AC biſects it. Wherefore likewiſe it ſhall be ‡ double to the Triangle EBC. *If*, therefore, *a Parallelogram and Triangle have both the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double the Triangle*; which was to be demonſtrated.

* 37 *of this.*
† 34 *of this.*
‡ *Ax.* 6.

PROPOSITION XLII.

PROBLEM.

To conſtitute a Parallelogram equal to a given Triangle, in an Angle equal to a given Right-lined Angle.

LET the given Triangle be ABC, and the Right-lined Angle D. It is required to conſtitute a Parallelogram equal to the given Triangle ABC, in a Right-lined Angle equal to D.

* 10 of this. Bisect * BC in E, join AE, and at the Point E,
† 23 of this. in the Right Line EC, constitute † an Angle CEF
‡ 31 of this. equal to D. Also draw ‡ AG thro' A, parallel to EC, and thro' C the Right Line CG, parallel to FE.

Now FECG is a Parallelogram: And because
* 38 of this. BE is equal to EC, the Triangle ABE shall be * equal to the Triangle AEC; for they stand upon equal Bases BE, EC, and are between the same Parallels BC, AG. Wherefore the Triangle ABC is double to the Triangle AEC. But the Parallelogram FECG
† 41 of this. is also † double to the Triangle AEC; for it has the same Base, and is between the same Parallels. Therefore the Parallelogram FECG is equal to the Triangle ABC, and has the Angle CEF equal to the Angle D. Wherefore, *the Parallelogram* FECG *is constituted equal to the given Triangle* ABC, *in an Angle* CEF *equal to a given Angle* D; which was to be done.

PROPOSITION XLIII.

THEOREM.

In every Parallelogram, the Complements of the Parallelograms, that stand about the Diameter, are equal between themselves.

LET ABCD be a Parallelogram, whose Diameter is DB; and let FH, EG, be Parallelograms standing about the Diameter BD. Now AK, KC, are called the Complements of them: I say, the Complement AK is equal to the Complement KC.

For since ABCD is a Parallelogram, and BD is
* 34 of this. the Diameter thereof, the Triangle ABD is * equal to the Triangle BDC. Again, because HKFD is a Parallelogram, whose Diameter is DK, the Triangle HDK shall * be equal to the Triangle DFK; and for the same Reason the Triangle KBG is equal to the Triangle KEB. But since the Triangle BEK is equal to the Triangle BGK, and the Triangle HDK to DFK; the Triangle BEK, together with the Triangle HDK, is equal to the Triangle BGK, together with the Triangle DFK. But the whole Triangle ABD is likewise equal to the whole Triangle BDC.

BDC. Wherefore the Complement remaining, AK, will be equal to the remaining Complement KC. Therefore, *in every Parallelogram, the Complements of the Parallelograms that stand about the Diameter, are equal between themselves*; which was to be demonstrated.

PROPOSITION XLIV.

PROBLEM.

To apply a Parallelogram to a given Right Line, equal to a given Triangle, in a given Right-lined Angle.

LET the Right Line given be AB, the given Triangle C, and the given Right-lined Angle D. It is required to the given Right Line AB, to apply a Parallelogram equal to the given Triangle C, in an Angle equal to D.

Make the Parallelogram BEFG equal to the * Triangle C, in the Angle EBG, equal to D. Place BE in a strait Line with AB, and produce FG to H, and through A let AH be drawn † parallel to either GB, or FE, and join HB. * 42 *of this.* † 31 *of this.*

Now, because the Right Line HF falls on the Parallels AH, EF, the Angles AHF, HFE, are ‡ equal to two Right Angles. And so BHF, HFE, are less than two Right Angles; but Right Lines making less than two Right Angles, with a third Line, being infinitely produced, will meet * each other. Wherefore HB, FE, produced, will meet each other; which let be in K, through which * draw KL parallel to EA, or FH, and produce AH, GB, to the Points L and M. ‡ 29 *of this.* * *Ax.* 12. * 31 *of this.*

Therefore HLKF is a Parallelogram, whose Diameter is HK; and AG, ME, are Parallelograms about HK; whereof LB, BF are the Complements. Therefore LB is † equal to BF. But BF is also equal to the Triangle C. Wherefore likewise LB shall be equal to the Triangle C; and because the Angle GBE is ‡ equal to the Angle ABM, and also equal to the Angle D, the Angle ABM shall be equal to the Angle D. Therefore, *to the given Right Line* AB *is applied a Parallelogram, equal to the given Triangle* C, *and the Angle* ABM, *equal to the given Angle* D; which was to be done. † 43 *of this.* ‡ 15 *of this.*

PRO-

PROPOSITION XLV.

PROBLEM.

To make a Parallelogram equal to a given Right-lined Figure, in a given Right-lined Angle.

LET ABCD be the given Right-lined Figure, and E the Right-lined Angle given. It is required to make a Parallelogram equal to the Right-lined Figure ABCD, in an Angle equal to E.

* of this. Let DB be joined, and make * the Parallelogram FH equal to the Triangle ADB, in an Angle HKF, equal to the given Angle E.

† 44 of this. Then to the Right Line GH apply † the Parallelogram GM, equal to the Triangle DBC in an Angle GHM, equal to the Angle E.

Now, becaufe the Angle E is equal to HKF, or GHM, the Angle HKF fhall be equal to GHM, add KHG to both; and the Angles HKF, KHG, are, together, equal to the Angles KHG, GHM.

‡ 29 of this. But HKF, KHG, are ‡, together, equal to two Right Angles. Wherefore, likewife, the Angles KHG, GHM, fhall be equal to two Right Angles: And fo, at the given Point H in the Right Line GH, two Right Lines KH, HM, not drawn on the fame Side, make the adjacent Angles, both together, equal to

* 14 of this. two Right Angles; and confequently KH, HM *, make one ftrait Line. And becaufe the Right Line HG falls upon the Parallels KM, FG, the alternate Angles MHG, HGF, are ‡ equal. And if HGL be added to both, the Angles MHG, HGL, together, are equal to the Angles HGF, HGL, together.

* 29 of this. But the Angles MHG, HGL, are * together equal to two Right Angles. Wherefore, likewife, the Angles HGF, HGL, are together equal to two Right Angles; and fo, FG, GL, make one ftrait Line. And fince KF is equal and parallel to HG, as like-

† 30 of this. wife HG to ML, KF fhall be † equal and parallel to ML, and the Right Lines KM, FL, join them.

‡ 33 of this. Wherefore KM, FL, are ‡ equal and parallel. Therefore KFLM is a Parallelogram. But fince the Triangle ABD is equal to the Parallelogram HF, and the

the Triangle DBC to the Parallelogram GM; therefore the whole Right-lined Figure ABCD will be equal to the whole Parallelogram KFLM. Therefore, *the Parallelogram* KFLM *is made equal to the given Right-lined Figure* ABCD, *in an Angle* FKM, *equal to the given Angle* E; which was to be done.

Coroll. It is manifest from what has been said, how to apply a Parallelogram to a given Right Line, equal to a given Right-lined Figure in a given Right-lined Angle.

PROPOSITION XLVI.

PROBLEM.

To describe a Square upon a given Right Line.

LET AB be the Right Line given, upon which it is required to describe a Square.

Draw * AC at Right Angles to AB from the Point A given therein; make † AD equal to AB, and thro' the Point D draw ‡ DE parallel to AB; also thro' B draw BE parallel to AD. * *11 of this.* † *3 of this.* ‡ *31 of this.*

Then ADEB is a Parallelogram; and so AB * is equal to DE, and AB to BE. But BA is equal to AD. Therefore the four Sides BA, AD, BE, ED, are equal to each other. * *34 of this.*

And so the Parallelogram ADEB is equilateral: I say, it is likewise equiangular. For, because the Right Line AD falls upon the Parallels AB, DE, the Angles BAD, ADE, are † equal to two Right Angles. But, BAD is a Right Angle: Wherefore ADE is also a Right Angle; but the opposite Sides and opposite Angles of Parallelograms are ‡ equal. Therefore, each of the opposite Angles ABE, BED, are Right Angles; and consequently ADBE is a Rectangle: But it has been proved to be equilateral. Therefore, *it is necessarily a * Square, and is described upon the Right Line* AB; which was to be done. † *29 of this.* ‡ *34 of this.* * *Def. 30.*

Coroll. Hence every Parallelogram that has one Right Angle, is a Rectangle.

PRO-

PROPOSITION XLVII.

THEOREM.

In any Right-angled Triangle, the Square described upon the Side, subtending the Right Angle, is equal to both the Squares, described upon the Sides, containing the Right Angle.

LET ABC be a Right-angled Triangle, having the Right Angle B A C. I say, the Square, described upon the Right Line B C, is equal to both the Squares, described upon the Sides B A, A C.

* 46 of this. For, describe * upon B C the Square B D E C, and on BA, AC, the Squares GB, HC; and through the

‖ 31 of this. Point A draw A L parallel to ‖ BD, or C E; and let A D, F C, be joined.

† Def. 30. Then, because the Angle BAC, BAG, † are Right ones, two Right Lines A G, A C, at the given Point A, in the Right Line B A, being on contrary Sides thereof, making the adjacent Angles equal to two Right

‡ 14 of this. Angles. Therefore C A, A G, make ‡ one strait Line, by the same Reason A B, A H, make one strait Line. And since the Angle D B C is equal to the Angle F B A, for each of them is a Right one, add A B C, which is common, and the whole Angle

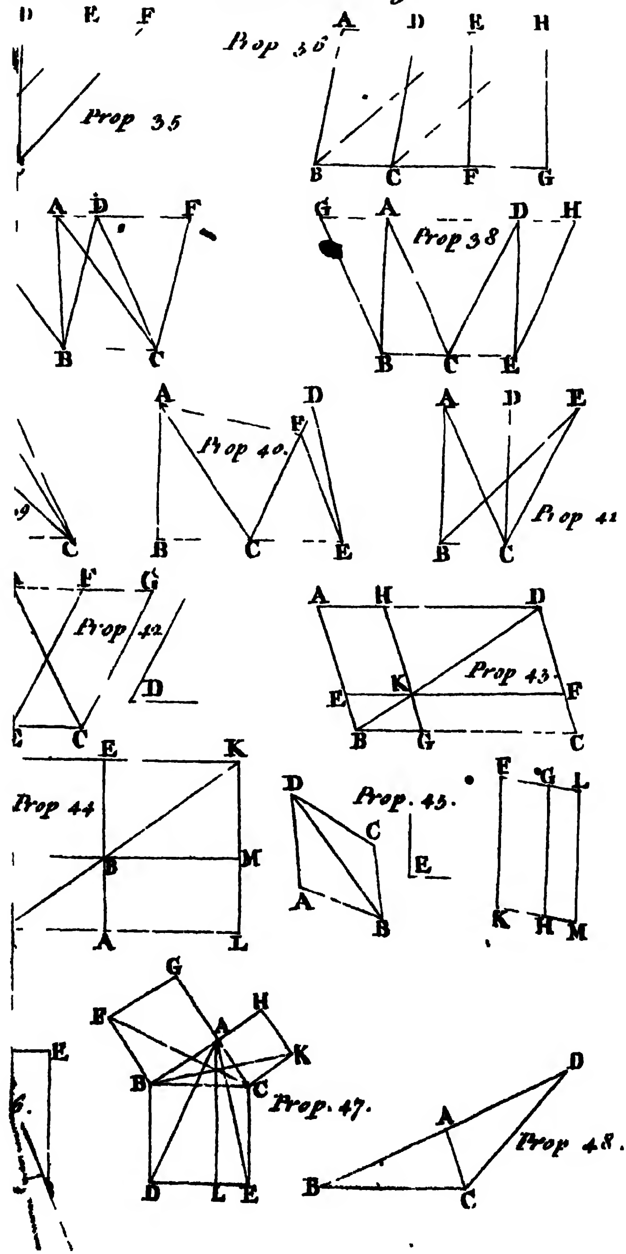
* Ax. 2. D B A is * equal to the whole Angle F B C. And since the two Sides A B, B D, are equal to the two Sides F B, B C, each to each, and the Angle D B A

† 4 of this. equal to the Angle F B C; the Base A D will be † equal to the Base F C, and the Triangle A B D equal to the Triangle F B C: But the Parallelogram B L

‡ 41 of this. is ‡ double to the Triangle A B D; for they have the same Base D B, and are between the same Parallels B D, A L. The Square G B is ‡ also double to the Triangle F B C; for they have the same Base F B, and are in the same Parallels F B, G C. But Things

* Ax. 6. that are the Doubles of equal Things, are * equal to each other. Therefore the Parallelogram B L is equal to the Square G B. After the same manner, A E, B K, being joined, we prove that the Parallelogram C L is equal to the Square H C. Therefore the whole Square BDEC is equal to the two Squares GB, HC. But the Square BDEC is described on the Right Line

BC,


Prop 35
Prop 36
Prop 38
Prop 40.
Prop 41
Prop 42
Prop 43.
Prop 44
Prop. 45.
Prop. 47.
Prop 48.

BC, and the Squares GB, HC, on BA, AC. Therefore the Square BE, deſcribed on the Side BC, is equal to the Squares deſcribed on the Sides BA, AC. Wherefore, *in any Right-angled Triangle, the Square deſcribed upon the Side, ſubtending the Right Angle, is equal to both the Squares deſcribed upon the Sides, containing the Right Angle*; which was to be demonſtrated.

PROPOSITION XLVIII.

THEOREM.

If a Square deſcribed upon one Side of a Triangle, be equal to the Squares deſcribed upon the other two Sides of the ſaid Triangle; then, the Angle contained by thoſe two other Sides is a Right Angle.

IF the Square, deſcribed upon the Side BC of the Triangle ABC, be equal to the Squares, deſcribed upon the other two Sides of the Triangle BA, AC; I ſay, the Angle BAC is a Right one.

For, let there be drawn AD from the Point A, at Right Angles, to AC; Likewiſe make AD equal to BA, and join DC.

Then, becauſe DA is equal to AB, the Square deſcribed on DA will be equal to the Square deſcribed on AB. And adding the common Square deſcribed on AC, the Squares deſcribed on DA, AC, are equal to the Squares deſcribed on BA, AC. But the Square deſcribed on DC is * equal to the Squares deſcribed on DA, AC; for DAC is a Right Angle: But the Square on BC is put equal to the Squares on BA, AC. Therefore the Square deſcribed on DC is equal to the Square deſcribed on BC; and ſo the Side CD is equal to the Side CB. And becauſe DA is equal to AB, and AC is common, the two Sides DA, AC, are equal to the two Sides BA, AC; and the Baſe DC is equal to the Baſe CB. Therefore the Angle DAC is ‡ equal to the Angle BAC; but DAC is a Right Angle; and ſo BAC will be a Right Angle alſo. *If*, therefore, *a Square deſcribed upon one Side of a Triangle, be equal to the Squares, deſcribed upon the other two Sides of the ſaid Triangle, then the Angle, contained by theſe two other Sides, is a Right Angle*; which was to be demonſtrated.

* 47 *of this.*

‡ 8 *of this.*

EUCLID's

EUCLID's

ELEMENTS.

BOOK II.

DEFINITIONS.

I. *EVERY Right-angled Parallelogram is said to be contained under two Right Lines, comprehending a Right Angle.*

II. *In every Parallelogram, either of those Parallelograms, that are about the Diameter, together with the Complements, is called a Gnomon.*

PRO-

PROPOSITION I.

THEOREM.

If there be two Right Lines, and one of them be divided into any Number of Parts; the Rectangle comprehended under the whole Line and the divided Line, shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line.

LET A and BC be two Right Lines, whereof BC is cut or divided any how in the Points D, E. I say, the Rectangle contained under the Right Lines A and BC, is equal to the Rectangles contained under A and BD, A and DE, and A and EC.

For, let * BF be drawn from the Point B, at Right Angles, to BC; and make † BG equal to A; and let ‡ GH be drawn thro' G parallel to BC: Likewise let ‡ there be drawn DK, EL, CH, through D, E, C, parallel to BG.

* 11. 1.
† 3. 1.
‡ 31. 1.

Then the Rectangle BH, is equal to the Rectangles BK, DL, and EH; but the Rectangle BH is that contained under A and BC; for it is contained under GB, BC; and GB is equal to A; and the Rectangle BK is that contained under A and BD; for it is contained under GB and BD; and GB is equal to A; and the Rectangle DL is that contained under A and DE, because DK, that is, BG, is equal to A: So likewise the Rectangle EH is that contained under A and EC. Therefore the Rectangle under A and BC is equal to the Rectangles under A and BD, A and DE, and A and EC. Therefore, *if there be two Right Lines given, and one of them be divided into any Number of Parts, the Rectangle comprehended under the whole Line and the divided Line, shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line*; which was to be demonstrated.

PRO-

PROPOSITION II.

Theorem.

If a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.

LET the Right Line AB be any how divided in the Point C. I say, the Rectangle contained under AB and BC, together with that contained under AB, and AC, is equal to the Square made on AB.

* 46. 1. For let the Square ADEB be described * on AB, and thro' C let CF be drawn parallel to AD or BE. Therefore AE is equal to the Rectangles AF and CE. But AE is a Square described upon AB; and AF is the Rectangle contained under BA and AC; for it is contained under DA and AC, whereof AD is equal to AB; and the Rectangle CE is contained under AB and BC, since BE is equal to AB. Wherefore the Rectangle under AB and AC, together with the Rectangle under AB and BC, is equal to the Square of AB. Therefore, *if a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line*; which was to be demonstrated.

PROPOSITION III.

Theorem.

If a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts, together with the Square of the first-mentioned Part.

LET the Right Line AB be any how cut in the Point C. I say, the Rectangle under AB and BC is equal to the Rectangle under AC and BC, together with the Square described on BC.

For

For describe * the Square CDEB upon BC; produce ED to F; and let AF be drawn † thro' A, parallel to CD or BE. * 46. 1. † 31. 1.

Then the Rectangle AE shall be equal to the two Rectangles AD, CE: And the Rectangle AE is that contained under AB and BC; for it is contained under AB and BE, whereof BE is equal to BC: And the Rectangle AD is that contained under AC and CB, since DC is equal to CB: And DB is a Square described upon BC. Wherefore the Rectangle under AB and BC is equal to the Rectangle under AC and CB, together with the Square described upon BC. Therefore, *if a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts, together with the Square of the first-mentioned Part*; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments.

LET the Right Line AB be any how cut in C. I say, the Square made on AB is equal to the Squares of AC, CB, together with twice the Rectangle contained under AC, CB.

For * describe the Square ADEB upon AB, join BD, and thro' C darw † CGF parallel to AD or BE; and also thro' G draw HK parallel to AB or DE. * 46. 1. † 31. 1.

Then, because CF is parallel to AD, and BD falls upon them, the outward Angle BGC shall be ‡ equal to the inward and opposite Angle ADB; but the Angle ADB is * equal to the Angle ABD, since the Side BA is equal to the Side AD. Wherefore the Angle CGB is equal to the Angle GBC; and so the Side BC equal † to the Side CG; but likewise the Side CB is ‡ equal to the Side GK, and the Side CG to BK. Therefore GK is equal to KB, and CGKB ‡ 29. 1. * 5. 1. † 6. 1. ‡ 34. 1.

is equilateral. I say, it is also Right-angled; for, because CG is parallel to BK, and CB falls on them,
† 29. 1. the Angles KBC, GCB †, are equal to two Right Angles. But KBC is a Right Angle. Wherefore, GCB also is a Right Angle, and the opposite CGK, GKB, shall be Right Angles. Then CGKB is a Rectangle. But it has been proved equilateral. Therefore CGKB is a Square described upon BC. For the same Reason HF is also a Square
† 34. 1. made upon HG, and (because HG is equal to AC †) it is equal to the Square of AC. Wherefore HF and CK are the Squares of AC and CB. And, because
* 43. 1. the Rectangle AG is * equal to the Rectangle GE, and AG is that which is contained under AC and CB; for GC is equal to CB; therefore GE shall be equal to the Rectangle under AC, and CB. Wherefore the Rectangles AG, and GE, are equal to twice the Rectangle contained under AC, and CB; and HF and CK, are the Squares of AC, CB. Therefore the four Figures HF, CK, AG, GE, are equal to the Squares of AC and CB, with twice the Rectangle contained under AC and CB. But HF, CK, AG, GE, make up the whole Square of AB, *viz.* ADEB. Therefore the Square of AB is equal to the Squares of AC and CB, together with twice the Rectangle contained under AC and CB. Wherefore, *if a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments*; which was to be demonstrated.

Coroll. Hence it is manifest, that the Parallelograms which stand about the Diameter of a Square, are likewise Squares.

PRO-

PROPOSITION V.

THEOREM.

If a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line.

LET any Right Line AB be cut into two equal Parts in C, and into two unequal Parts in D. I say, the Rectangle contained under AD, and DB, together with the Square of CD, is equal to the Square of BC.

For † describe CEFB, the Square of BC; draw BE, and through D draw * DHG, parallel to CE, or BF; and through H draw KLO, parallel to CB, or EF; and AK through A, parallel to CL, or BO. † 46. 1. * 31. 1.

Now the Complement CH is ‡ equal to the Complement HF. Add DO, which is common to both of them, and the whole CO is equal to the whole DF: But CO is equal to AL, because AC is equal to CB †; therefore AL is equal to DF; and, adding CH, which is common, the whole AH shall be equal to FD, DL, together. But AH is the Rectangle contained under AD, and DB; for DH is * equal to DB, and FD, DL, is the Gnomon MNX; therefore MNX is equal to the Rectangle contained under AD, and DB; and if LG, being common, and equal to the Square of CD, be added, then the Gnomon MNX, and LG, are equal to the Rectangle contained under AD, and DB, together with the Square of CD; but the Gnomon, MNX, and LG, make up the whole Square CEFB, *viz.* the Square of CB. Therefore the Rectangle under AD, and DB, together with the Square of CD, is equal to the Square of CB. Wherefore, *if a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line*; which was to be demonstrated. ‡ 43. 1. † 36. 1. * Cor. 4. of this.

PROPOSITION VI.

THEOREM.

If a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contained under [the Line compounded of] the whole and added Line (taken as one Line) and the added Line, together with the Square of half the first Line, is equal to the Square of [the Line compounded of] half the Line and the added Line, taken as one Line.

LET the Right Line AB be bisected in the Point C, and BD added directly thereto. I say, the Rectangle under AD, and BD, together with the Square of BC, is equal to the Square of CD.

* 46. 1. For, describe * CEFD, the Square of CD, and
† 31. 1. join DE; draw † BHG thro' B, parallel to CE, or DF, and KLM thro' H, parallel to AD, or EF, as also AK thro' A, parallel to CL, or DM.

Then because AC is equal to CB, the Rectangle
‡ 36. 1. AL shall be ‡ equal to the Rectangle CH; but CH
* 43. 1. is * equal to HF. Therefore AL will be equal to HF; and adding CM, which is common to both, then the whole Rectangle AM is equal to the Gnomon NXO. But AM is that Rectangle which is
† Cor. 4. of this. contained under AD, and DB; for DM is † equal to DB; therefore the Gnomon NXO is equal to the Rectangle under AD, and DB. Add LG, which
‡ Cor. 4. of this. is common, *viz.* ‡ the Square of CB; and then the Rectangle under AD, DB, together with the Square of BC, is equal to the Gnomon NXO with LG. But the Gnomon NXO, and LG, together, make up the Figure CEFD, that is, the Square of CD. Therefore the Rectangle under AD, and DB, together with the Square of BC, is equal to the Square of CD. Therefore, *if a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contained under [the Line compounded of] the whole and added Line (taken as one Line) and the*

added

added Line, together with the Square of half the first Line, is equal to the Square of [the Line compounded of] half the Line and the added Line, taken as one Line; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the ſaid Segment, together with the Square made of the other Segment.

LET the Right Line A B be any how cut in the Point C. I ſay, the Squares of A B, and B C, together, are equal to double the Rectangle contained under A B, and B C, together with the Square, made of A C.

For let the Square of A B be * deſcribed, *viz.* A D E B, and conſtruct § the Figure. * 46. 1.

Then, becauſe the Rectangle A G is † equal to the Rectangle GE; if CF, which is common, be added to both, the whole Rectangle A F ſhall be equal to the whole Rectangle CE; and ſo the Rectangles AF, CE, taken together, are double to the Rectangle AF; but AF, CE, make up the Gnomon KLM, and the Square CF. Therefore the Gnomon K L M, together with the Square C F, ſhall be double to the Rectangle A B. But double the Rectangle under AB, and BC, is double the Rectangle AF; for BF is ‡ equal to BC. Therefore the Gnomon KLM, and the Square CF, are equal to twice the Rectangle contained under AB, and BC. And if H F, which is common, being the Square of AC, be added to both, then the Gnomon KLM, and the Squares CF, HF, are equal to double the Rectangle † 43. 1. ‡ Cor. 4.

§ *A Figure is ſaid to be conſtructed, when Lines drawn in a Parallelogram, parallel to the Sides thereof, cut the Diameter in one Point; and make two Parallelograms about the Diameter, and two Complements. So likewiſe a double Figure is ſaid to be conſtructed, when two Right Lines, parallel to the Sides, make four Parallelograms about the Diameter, and four Complements.*

contained under AB, and BC, together with the Square of AC. But the Gnomon KLM, together with the Squares CF, HF, are equal to ADEB, and CF, *viz.* the Squares of AB, BC. Therefore the Squares of AB, and BC, are together equal to double the Rectangle contained under AB, and BC, together with the Square of AC. Therefore, *if a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the said Segment, together with the Square, made of the other Segment*; which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If a Right Line be any how cut into two Parts, four Times the Rectangle, contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of [the Line compounded of] the whole Line and the first Part, taken as one Line.

LET the Right Line AB be cut any how in C. I say, four Times the Rectangle contained under AB, and BC, together with the Square of AC, is equal to the Square of AB, and BC, taken as one Line.

For, let the Right Line AB be produced to D, so that BD be equal to BC; describe the Square AEFD, on AD, and construct the double Figure.

* *Hyp.* † 34. 1. Now, since CB is * equal to BD, and also to † GK, and BD is equal to KN; therefore GK shall be likewise equal to KN: By the same Reasoning, PR is equal to RO. And since CB is equal to BD, and GK to KN, the Rectangle CK will ‡ be equal to the Rectangle BN, and the Rectangle GR to the Rectangle RN. But CK is ‖ equal to RN; for they are the Complements of the Parallelogram CO. Therefore BN is equal to GR, and the four Squares BN, CK, GR, RN, are equal to each other; and so they are together quadruple CK. Again, because CB is equal to BD, and BD to BK, that is, equal to CG; and the

‡ 36. 1.

‖ 43. 1.

said CB is equal also to GK, that is, to GP; therefore CG shall be equal to GP. But PR is equal to RO; therefore the Rectangle AG shall be equal to the Rectangle MP, and the Rectangle PL equal to RF. But MP is equal to PL; for they are the Complements of the Parallelogram ML. Wherefore AG is equal also to RF. Therefore the four Parallelograms AG, MP, PL, RF, are equal to each other, and accordingly they are together quadruple of AG. But it has been proved, that the four Squares CK, BN, GR, RN, are quadruple of CK. Therefore the four Rectangles, and the four Squares, making up the Gnomon STY, are together quadruple of AK; and because AK is a Rectangle contained under AB, and BC, for BK is equal to BC; therefore four Times the Rectangle under AB and BC, will be quadruple of AK. But the Gnomon STY has been proved to be quadruple of AK. And so four Times the Rectangle contained under AB, and BC, is equal to the Gnomon STY. And if XH, being equal to † the Square of AC, which is common, be added to both; then four Times the Rectangle contained under AB, and BC, together with the Square of AC, is equal to the Gnomon STY, and the Square XH. But the Gnomon STY and XH make AEFD, the whole Square of AD. Therefore four Times the Rectangle contained under AB, and BC, together with the Square of AC, is equal to the Square of AD, that is, of AB and BC taken as one Line. Wherefore, *if a Right Line be any how cut into two Parts, four Times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of [the Line compounded of] the whole Line and the first Part, taken as one Line*; which was to be demonstrated.

† *Cor. 4. of this.*

PRPOSITION IX.

THEOREM.

If a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts, together, are double to the Square of the half Line, and the Square of the intermediate Part.

LET any Right Line A B be cut unequally in D, and equally in C. I say, the Squares of AD, DB, together, are double to the Squares of A C and C D together.

* 11. 1. For, let * C E be drawn from the Point C at Right Angles to AB, which make equal to AC, or CB; and † 31. 1. join EA, EB. Also through D let † DF be drawn parallel to CE, and FG through F parallel to AB, and draw AF.

Now, because AC is equal to CE, the Angle EAC ‡ 5. 1. will be ‡ equal to the Angle AEC; and since the Angle at C is a Right one, the other Angles, A E C, E A C, * Cor. 3. 32. 1. together, shall * make one Right Angle, and are equal to each other: And so AEC, EAC, are each equal to half a Right Angle. For the same Reasons are also CEB, EBC, each of them half a Right Angle. Therefore the whole Angle A E B is a Right Angle. And since the Angle GEF is half a Right one, and EGF is † 29. 1. a Right Angle; for it is † equal to the inward and opposite Angle ECB; the other Angle EFG will be also equal to half a Right one. Therefore the Angle GEF ‡ 6. 1. is equal to the Angle EFG. And so the Side EG is ‡ equal to the Side GF. Again, because the Angle at B is half a Right one, and FDB is a Right one, because equal to the inward and opposite Angle ECB, the other Angle B F D will be half a Right Angle. Therefore the Angle at B is equal to the Angle BFD; and so the Side DF is equal to the Side DB. And because AC is equal to C E, the Square of A C will be equal to the Square of CE. Therefore the Squares of AC, and CE, together, are double to the Square of A C; but the † 47. 1. Square of E A is † equal to the Squares of A C, and

CE, together, since ACE is a Right Angle. Therefore the Square of EA is double to the Square of AC. Again, because EG is equal to GF, and the Square of EG is equal to the Square of GF; therefore the Squares of EG, GF, together, are double to the Square of GF. But the Square of EF is † equal to the Squares † 47. 1. of EG, GF. Therefore the Square of EF is double the Square of GF. But GF is equal to CD; and so the Square of EF is double to the Square of CD. But the Square of AE is likewise double to the Square of AC. Wherefore the Squares of AE, and EF, are double to the Squares of AC, and CD. But the Square of AF is † equal to the Squares of AE, and EF; because the Angle AEF is a Right Angle, and consequently the Square of AF is double to the Squares of AC, and CD. But the Squares of AD, and DF, are equal to the Square of AF: For the Angle at D is a Right Angle. Therefore the Squares of AD, and DF, together, shall be double to the Squares of AC, and CD, together. But DF is equal to DB. Therefore the Squares of AD, and DB, together, will be double to the Squares of AC, and CD, together. Wherefore *if a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part*; which was to be demonstrated.

PROPOSITION X.

THEOREM.

If a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.

LET the Right Line AB be bisected in C, and any strait Line BD added directly thereto. I say, the Squares of AD, and DB, together, are double to the Squares of AC, and CD, together.

For,

* 11. 1. For, draw * CE from the Point C at Right Angles to A B, which make equal to A C, or C B; and draw
† 31. 1. A E, E B; likewise through E let E F be † drawn parallel to A D, and through D, D F † parallel to C E.

Then, because the Right Line E F falls upon the
‡ 29. 1. Parallels EC, FD, the Angles CEF, EFD, are ‡ equal to two Right Angles. Therefore the Angles F E B, EFD, are together less than two Right Angles. But Right Lines making, with a third Line, Angles together less than two Right Angles, being infinitely pro-
* Ax. 12. duced, will meet *. Wherefore EB, FD, produced, will meet towards B D. Now let them be produced, and meet each other in the Point G, and let A G be drawn.

And then, because A C is equal to C E, the Angle
† 5. 1. A E C will be equal to the Angle E A C †: But the Angle at C is a Right Angle. Therefore the Angle E A C, or A E C, is half a Right one. By the same Way of Reasoning, the Angle CEB, or EBC, is half a Right one. Therefore A E B is a Right Angle.
‡ 15. 1. And since EBC is half a Right Angle, DBG will ‡ also be half a Right Angle, since it is vertical to E B C.
* 29. 1. But B D G is a Right Angle also; for it is * equal to the alternate Angle D C E. Therefore the remaining Angle D G B is half a Right Angle, and so equal to
† 6. 1. D B G. Wherefore the Side B D is † equal to the Side D G. Again, because E G F is half a Right Angle, and the Angle at F is a Right Angle, for it is
* 34. 1. equal * to the opposite Angle at C; the remaining Angle FEG will be also half a Right one, and is equal
† 6. 1. to the Angle E G F; and so the Side G F is † equal to the Side E F. And since E C is equal to C A, and the Square of E C equal to the Square of C A; therefore the Squares of E C, C A, together, are double to
‡ 47. 1. the Square of CA. But the Square of EA is ‡ equal to the Squares of EC and CA. Wherefore the Square of E A is double to the Square of A C. Again, because G F is equal to F E, the Square of G F also is equal to the Square of F E. Wherefore the Squares of GF and FE are double to the Square of FE. But the Square of EG is ‡ equal to the Squares of GF, FE. Therefore the Square of E G is double to the Square of EF: But E F is equal to CD. Wherefore the Square of E G shall be double to the Square of C D.

But

But the Square of EA has been proved to be double to the Square of AC. Therefore the Squares of AE and EG, are double the Squares of AC and CD. But the Square of AG is † equal to the Squares of AE and EG, † 47. 1.
for the Angle (AEB or) AEG has been proved to be Right; and consequently the Square of AG is double to the Squares of AC, and CD. But the Squares of AD, and DG, are † equal the Square of AG. Therefore the Squares of AD, and DG, are double to the Squares of AC, and CD. But DG is equal to DB. Wherefore the Squares of AD, and DB, are double to the Squares of AC, and CD. Therefore, *if a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line*; which was to be demonstrated.

PROPOSITION XI.

PROBLEM.

To cut a given Right Line so, that the Rectangle contained under the whole Line and one Segment, shall be equal to the Square of the other Segment.

LET AB be a given Right Line. It is required to cut the same so, that the Rectangle contained under the Whole, and one Segment thereof, be equal to the Square of the other Segment.

Describe * ABDC the Square of AB; bisect AC in * 46. 1.
E, and draw BE: Also produce CA to F, so that EF be equal to EB. Describe * FGHA the Square of AF, and produce GH to K. I say, AB is cut in H so, that the Rectangle under AB and BH is equal to the Square of AH.

For since the Right Line AC is bisected in E, and AF is directly added thereto, the Rectangle under CF and FA, together with the Square of AE, will be † equal † 6 of this.
to the Square of EF. But EF is equal to EB. Therefore the Rectangle under CF and FA, together with the Square of AE, is equal to the Square of EB. But the

Squares

* -- 1. Squares of BA and AE are ‡ equal to the Square of EB; for the Angle at A is a Right Angle. Therefore the Rectangle under CF and FA, together with the Square of AE, is equal to the Squares of BA, and AE. And, taking away the Square of AE, which is common, the remaining Rectangle under CF and FA is equal to the Square of AB. But FK is the Rectangle under CF and FA; since AF is equal to FG; and the Square of AB is AD. Wherefore the Rectangle FK is equal to the Square AD. And if AK, which is common, be taken from both, then the remaining Square FH is equal to the remaining Rectangle HD. But HD is the Rectangle under AB and BH, since AB is equal to BD; and FH is the Square of AH. Therefore, *the Rectangle under* AB *and* BH *shall be equal to the Square of* AH. *And so the given Right Line* AB *is cut in* H, *so that the Rectangle under* AB *and* BH *is equal to the Square of* AH; which was to be done.

PROPOSITION XII.

THEOREM.

In an obtuse-angled Triangle, the Square of the Side subtending the obtuse Angle is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides, containing the obtuse Angle, viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular and the obtuse Angle.*

* 12. 1. LET ABC be an obtuse-angled Triangle, having the obtuse Angle BAC; and * from the Point B draw BD perpendicular to the Side CA produced. I say, the Square of BC is greater than the Squares of BA, and AC, by twice the Rectangle contained under CA, and AD.

† 4 *of this.* For, because the Right Line CD is any how cut in the Point A, the Square of CD shall be † equal to the Squares of CA, and AD, together with twice the Rectangle under CA, and AD. And if the Square of BD, which is common, be added, then the Squares of CD and DB are equal to the Squares of CA, AD,

and

and DB, and twice the Rectangle contained under CA and AD. But the Square of CB is * equal to the Squares of CD, DB; for the Angle at D is a Right one, since BD is perpendicular; and the Square of AB is * equal to the Squares of AD and DB. Therefore the Square of CB is equal to the Squares of CA, and AB, together with twice a Rectangle under CA, and AD. Therefore *in an obtuse-angled Triangle, the Square of the Side subtending the obtuse Angle is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides containing the obtuse Angle,* viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the Perpendicular and the obtuse Angle;* which was to be demonstrated. * 47. 1.

PROPOSITION XIII.

THEOREM.

In an acute-angled Triangle, the Square of the Side subtending the acute Angle is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. *that on which the Perpendicular falls, and the Line assumed within the Triangle, from the Perpendicular to the acute Angle.*

LET ABC be an acute-angled Triangle, having the acute-angle B; and from A let there * be drawn AD perpendicular to BC. I say, the Square of AC is less than the Squares of CB, and BA, by twice a Rectangle under CB, and BD. * 12. 1.

For, because the Right Line CB is cut any how in D, the Squares of CB, and BD will be † equal to twice a Rectangle under CB, and BD, together with the Square of DC. And if the Square of AD be added to both, then the Squares of CB, BD, and DA, are equal to twice the Rectangle contained under CB, and BD, together with the Squares of AD, and DC. But the Square of AB is ‡ equal to the Squares of BD, and DA; for the Angle at D is a Right Angle. And the Square of AC is ‡ equal to the † 7 *of this.* ‡ 47. 1.

 the

the Squares of AD and DC. Therefore the Squares of CB and BA are equal to the Square of AC, together with twice the Rectangle contained under CB and BD. Wherefore the Square of AC, only, is less than the Squares of CB and BA, by twice the Rectangle under CB and BD. Therefore, *in an acute-angled Triangle, the Square of the Side subtending the acute Angle is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle,* viz. *that on which the Perpendicular falls, and the Line assumed within the Triangle, from the Perpendicular to the acute Angle;* which was to be demonstrated.

PROPOSITION XIV.

PROBLEM.

To make a Square equal to a given Right-lined Figure.

LET A be the given Right-lined Figure. It is required to make a Square equal thereto.

* 45. 1. Make * the Right-angled Parallelogram BCDE equal to the Right-lined Figure A. Now if BE be equal to ED, what was proposed will be already done, since the Square BD is made equal to the Right-lined Figure A: But if it be not, let either BE or ED be the greater: Suppose BE, which let be produced to F; so that EF be equal to ED. This being done, let BF

† 10. 1. be † bisected in G, about which, as a Centre, with the Distance GB, or GF, describe the Semicircle BHF; and let DE be produced to H, and draw GH. Now, because the Right Line BF is divided into two equal Parts in G, and into two unequal ones in E, the Rectangle under BE and EF, together with the Square of

‡ 5 of this. GE, shall be ‡ equal to the Square of GF. But GF is equal to GH. Therefore the Rectangle under BE and EF, together, with the Square of GE, is equal to the Square of GH. But the Squares of HE and GE are

* 47. 1. * equal to the Square of GH. Wherefore the Rectangle under BE and EF, together with the Square of GE, is equal to the Squares of HE and GE. And if the Square

of EG, which is common, be taken from both, the remaining Rectangle, contained under BE and EF, is equal to the Square of EH. But the Rectangle under BE and EF is the Parallelogram BD, becauſe EF is equal to ED. Therefore the Parallelogram BD is equal to the Square of EH; but the Parallelogram BD is equal to the Right-lined Figure A. Wherefore the Right-lined Figure A is equal to the Square of EH. And ſo, *there is a Square made equal to the given Right-lined Figure* A, viz. *the Square of* EH; which was to be done.

The END *of the* SECOND BOOK.

EUCLID's ELEMENTS.

BOOK III.

DEFINITIONS.

I. *EQUAL Circles are such whose Diameters are equal; or from whose Centres the Right Lines that are drawn are equal.*

II. *A Right Line is said to touch a Circle, when meeting the same, and being produced, it does not cut it.*

III. *Circles are said to touch each other, which meeting do not cut one another.*

IV. *Right Lines in a Circle are said to be equally distant from the Centre, when Perpendiculars drawn from the Centre to them are equal.*

V. *And that Line is said to be farther from the Centre, on which the greater Perpendicular falls.*

VI. *A Segment of a Circle is a Figure contained under a Right Line, and a Part of the Circumference of a Circle.*

VII. *An Angle of a Segment is that which is contained by a Right Line, and the Circumference of a Circle.*

VIII. *An Angle is said to be in a Segment, when some Point is taken in the Circumference thereof, and from it Right Lines are drawn to the Ends of that Right Line, which is the Base of the Segment; then the Angle contained under the Lines, so drawn, is said to be an Angle in a Segment.*

IX. *But when the Right Lines containing the Angle do receive any Circumference of the Circle, then the Angle is said to stand upon that Circumference.*

X. *A Sector of a Circle is that Figure, which is comprehended between two Right Lines, drawn from the Centre, and the Circumference contained between them.*

XI. *Similar Segments of Circles are those, which include equal Angles, or whereof the Angles in them are equal.*

PROPOSITION I.

PROBLEM.

To find the Centre of a Circle given.

LET ABC be the Circle given. It is required to find the Centre thereof.

Let the Right Line AB be any how drawn, in it, which * bisect in the Point D; and let DC be † drawn from the Point D, at Right Angles to AB, which let be produced to E. * 10. 1. † 11. 1.

Then, if EC be * bisected in F, I say, the Point F is the Centre of the Circle ABC.

For, if it be not, let G be the Centre, and let GA, GD, GB, be drawn. Now, because DA is equal to DB, and DG is common, the two Sides AD, DG, are equal to the two Sides GD, DB, each to each; also the Base GA is ‡ equal to the Base GB, for they are drawn from the Centre G. Therefore the Angle ADG is * equal to the Angle GDB. But when a Right Line standing upon a Right Line makes the adjacent Angles equal to one another, each of the equal Angles will † be a Right Angle. Wherefore ‡ Def. 15. 1. * 8. 1. † Def. 10. 1.

the Angle GDB is a Right Angle. But [illegible] is also a Right Angle. Therefore, the Angle FDB is equal to the Angle GDB, a greater to a less, which is absurd. Wherefore G is not the Centre of the Circle ABC. After the same manner we prove, that no other Point, unless F, is the Centre. Therefore, F *is the Centre of the Circle* ABC; which was to be found.

Coroll. If in a Circle any Right Line cuts another Right Line into two equal Parts and at Right Angles, the Centre of the Circle will be in that cutting Line.

PROPOSITION II.

THEOREM.

If any two Points be assumed in the Circumference of a Circle, the Right Line joining those two Poi[illegible] shall fall within the Circle.

LET ABC be a Circle; in the Circumference of which let any two Points A, B, be assumed. I say, a Right Line drawn, from the Point A, to the Point B, falls within the Circle.

* 1 *of this.* Find D * the Centre of the given Circle, and let any Point E be taken in the Right Line AB, and let DA, DE, DB, be joined.

Then because DA is equal to DB, the Angle DAB
† 5. 1. will be † equal to the Angle DBA; and since the Side AE of the Triangle DAE is produced, the Angle DEB
‡ 16. 1. will be ‡ greater than the Angle DAE. But the Angle DAE is equal to the Angle DBE; therefore the Angle DEB is greater than the Angle DBE. But the greater
‖ 19. 1. Angle subtends the greater Side. Wherefore DB ‖ is greater than DE. But DB only comes to the Circumference of the Circle; therefore DE does not reach so far. And so the Point E falls within the Circle. Therefore, *if two Points are assumed in the Circumference of a Circle, the Right Line joining those two Points shall fall within the Circle*; which was to be demonstrated.

Coroll. Hence if a Right Line touches a Circle, it will touch it in one Point only.

PRO-

PROPOSITION III.

THEOREM.

If in a Circle, a Right Line drawn thro' the Centre cuts any other Right Line, not drawn thro' the Centre, into equal Parts, it shall cut it at Right Angles; and if it cuts it at Right Angles, it shall cut it into two equal Parts.

LET ABC be a Circle, wherein the Right Line CD, drawn thro' the Centre, bisects the Right Line AB, not drawn thro' the Centre. I say, it cuts it at Right Angles.

For, * find E the Centre of the Circle, and let EA, EB, be joined. * 1 *of this.*

Then because AF is equal to FB, and FE is common, the two Sides AF, FE, are equal to the two Sides BF, FE, each to each; but the Base [illegible] is equal to the Base EB. Wherefore the Angle AFE shall be † equal to the Angle BFE. But when a Right Line standing upon a Right Line makes the adjacent Angles equal to one another, each of the equal Angles is ‡ a Right Angle. Wherefore AFE, or BFE, is a Right Angle. And therefore the Right Line CD drawn thro' the Centre, bisecting the Right Line AB, not drawn thro' the Centre, cuts it at Right Angles. Now, if CD cuts AB at Right Angles, I say, it will bisect it; that is, AF will be equal to FB: For the same Construction remaining, because EA, being drawn from the Centre, is equal to EB, the Angle EAF shall be * equal to the Angle EBF. But the Right Angle AFE is equal to the Right Angle BFE: Therefore the two Triangles EAF, EBF, have two Angles of the one equal to two Angles of the other, and the Side EF is common to both. Wherefore the other Sides of the one shall be † equal to the other Sides of the other: And so AF will be equal to FB. Therefore, *if in a Circle, a Right Line drawn thro' the Centre cuts any other Right Line, not drawn thro' the Centre, into two equal Parts, it shall cut it at Right Angles; and if it cuts it at Right Angles, it shall cut it into two equal Parts*; which was to be demonstrated.

† 8. 1.
‡ *Def.* 10. 1.
* 5. 1.
† 26. 1.

PROPOSITION IV.

THEOREM.

If in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts.

LET ABCD be a Circle, wherein two Right Lines AC, BD, not drawn thro' the Centre, cut each other in the Point E. I say, they do not bisect each other.

For, if possible, let them bisect each other, so that AE be equal to EC, and BE to ED. Let the Centre
1 of this. F of the Circle ABCD be † found, and join EF.

Then, because the Right Line FE, drawn thro' the Centre, bisects the Right Line AC, not drawn thro' the Ce[redacted] it will * cut AC at Right Angles. And so FEA is a Right Angle. Again, because the Right Line FE, drawn thro' the Centre, bisects the Right Line BD, not drawn thro' the Centre, it will * cut BD at Right Angles. Therefore FEB is a Right Angle. But FEA has been shewn to be also a Right Angle. Wherefore the Angle FEA will be equal to the Angle FEB, a less to a greater; which is absurd. Therefore, AC, BD, do not mutually bisect each other. And so, *if in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other in two equal Parts*; which was to be demonstrated.

PROPOSITION V.

THEOREM.

If two Circles cut one another, they shall not have the same Centre.

LET the two Circles ABC, CDG, cut each other in the Points B, C. I say, they have not the same Centre.

For, if they have, let it be E, and join EC, and draw EFG at Pleasure.

Now, becauſe E is the Centre of the Circle A B C, CE will be equal to EF. Again, becauſe E is the Centre of the Circle CDG, CE is equal to the EG. But CE has been ſhewn to be equal to E F. Therefore E F ſhall be equal to EG, a leſs to a greater, which cannot be. Therefore the Point E is not the Centre of both the Circles A B C, C D G. Wherefore, *if two Circles cut one another, they ſhall not have the ſame Centre*; which was to be demonſtrated.

PROPOSITION VI.

Theorem.

If two Circles touch one another inwardly, they will not have one and the ſame Centre.

LET two Circles ABC, CDE, touch one another inwardly in the Point C. I ſay, they will not have one and the ſame Centre.

For, if they have, let it be F, and join FC, and draw FB any how.

Then, becauſe F is the Centre of the Circle ABC, CF is equal to FB. And becauſe F is alſo the Centre of the Circle CDE, CF ſhall be equal to FE. But CF has been ſhewn to be equal to F B. Therefore F E is equal to F B, a leſs to a greater; which cannot be. Therefore the Point F is not the Centre of both the Circles ABC, CDE. Wherefore, *if two Circles touch one another inwardly, they will not have one and the ſame Centre*; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If in the Diameter of a Circle some Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greatest of these Lines shall be that wherein the Centre of the Circle is, the least, the Remainder of the same Line. And of all the other Lines, the nearest to that which was drawn thro' the Centre, is always greater than that more remote; and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the least or greatest Line.

LET ABCD be a Circle, whose Diameter is AD, in which assume some Point F, which is not the Centre of the Circle. Let the Centre of the Circle be E; and from the Point F, let certain Right Lines FB, FC, FG, fall on the Circumference: I say, FA is the greatest of these Lines, and FD the least; and of the others FB is greater than FC, and FC greater than FG.

For, let BE, CE, GE, be joined.

Then, because two Sides of every Triangle are
* 20. 1. * greater than the third; BE and EF are greater than BF. But AE is equal to BE. Therefore BE and EF are equal to AF. And so AF is greater than FB.

Again, because BE is equal to CE, and FE is common, the two Sides BE and FE are equal to the two Sides CE and EF. But the Angle BEF is greater than the Angle CEF. Wherefore the Base BF is greater
† 24. 1. than the Base FC †. For the same Reason, CF is greater than FG.

‡ 20. 1. Again, because GF and FE are greater than GE, and GE is equal to ED; GF and FE shall be greater than ED; and if FE, which is common, be taken away, then the Remainder GF is greater than the Remainder FD. Wherefore, FA *is the greatest of the Right Lines, and* FD *the least: Also* BF *is greater than* FC, *and* FC *greater than* FG.

I say,

I ſay, moreover, that there are only two equal Right Lines, that can fall from the Point F on ABCD, the Circumference of the Circle on each Side the ſhorteſt Line F D. For at the given Point E, with the Right Line EF, make ‡ the Angle FEH equal to the Angle GEF, and join FH. Now becauſe GE is equal to EH, and EF is common, the two Sides GE and EF are equal to the two Sides HE and EF. But the Angle GEF is equal to the Angle HEF. Therefore the Baſe FG ſhall be † equal to the Baſe FH. I ſay, no other Right Line falling from the Point F, on the Circle, can be equal to FG. For if there can, let this be FK. Now, ſince FK is equal to FG, and FH is alſo equal to FG; therefore FK will be equal to FH, *viz.* a Line drawn nigher to that paſſing thro' the Centre, equal to one more remote, which * cannot be. *If, therefore, in the Diameter of a Circle, ſome Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greateſt of theſe Lines ſhall be that wherein the Centre of the Circle is; the leaſt, the Remainder of the ſame Line. And of all the other Lines, the neareſt to that which was drawn thro' the Centre, is always greater than that more remote; and only two equal Lines fall from the aboveſaid Point upon the Circumference, on each Side of the leaſt or greateſt Line;* which was to be demonſtrated.

‡ 23. 1.

† 4. 1.

* *by this.*

PROPOSITION VIII.

THEOREM.

If some Point be assumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which passes thro' the Centre, but the other any how; the greatest of the Lines which fall upon the concave Part of the Circumference of the Circle, is that passing thro' the Centre; and of the others, that which is nearest to the Line, passing thro' the Centre, is greater than that more remote. But the least of the Lines that fall upon the convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the least, is less than that which is farther distant; and from that Point there can be drawn only two equal Lines, which shall fall on the Circumference on each Side the least Line.

LET ABC be a Circle, out of which take any Point D. From this Point let there be drawn certain Right Lines DA, DE, DF, DC, to the Circle, whereof DA passes thro' the Centre. I say, DA, which passes thro' the Centre, is the greatest of the Lines falling upon AEFC, the concave Circumference of the Circle: Likewise DE is greater than DF, and DF greater than DC. But of the Lines that fall upon HLKG the convex Circumference of the Circle, the least is DG, *viz.* the Line drawn from D, to the Diameter GA; and that which is nearest the least DG, is always less than that more remote; that is, DK is less than DL, and DL less than [redacted]

* 1 of this. For, find * M the Centre of the Circle ABC, and let ME, MF, MC, MH, ML, MK, be joined.

Now, because AM is equal to EM; if MD, which is common, be added, AD will be equal to EM and

† 20. 1. MD. But EM and MD are † greater than ED;

there-

therefore AD is also greater than ED. Again, because ME is equal to MF, and MD is common, then ME and MD shall be equal to MF and MD; but the Angle EMD is greater than the Angle FMD. Therefore the Base ED will be † greater than the Base FD. † 24. 1.
We prove, in the same manner, that FD is greater than CD. Wherefore, DA *is the* [illegible]*test of the Right Lines falling from the Point* D; [illegible] *is greater than* DF, *and* DF *is greater than* DC.

Moreover, because MK and KD are * greater than MD, and MG is equal to MK; then the Remainder KD will † be greater than the Remainder GD. And so GD is less than KD, and consequently is the least. And because two Right Lines MK, KD, are drawn from M and D to the Point K, within the Triangle MLD, MK, and KD, are ‡ less than ML and LD; but MK is equal to ML. Wherefore the Remainder DK is less than the Remainder DL. In like manner we demonstrate, that DL is less than DH. Therefore, DG *is the least*; *and* DK *is less than* DL, *and* DL *than* DH.

* 20. 1.
† Ax. 4.
‡ 21. 1.

I say likewise, that from the Point D only two equal Right Lines can fall upon the Circle on each Side the least Line. For, make * the Angle DMB at the Point M, with the Right Line MD, equal to the Angle KMD, and join DB. Then, because MK is equal to MB, and MD is common, the two Sides KM, MD, are equal to the two Sides MB, MD, each to each; but the Angle KMD is equal to the Angle BMD. Therefore the Base DK is † equal to the Base DB. Now I say, no other Line can be drawn from the Point D to the Circle equal to DK, for, if there can, let it be DN. Now, since DK is equal to DN, as also to DB, therefore DB shall be equal to DN, *viz.* the Line drawn nearest to the least equal to that more remote, which has been * shewn to be impossible. Therefore, *if some Point be assumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which passes thro'* [illegible] *but the others any how; the greatest of the Lines, that fall upon the concave Part of the Circumference of the Circle, is that passing thro' the Centre; and of the others, that which is nearest to the Line, passing thro' the Centre, is greater than that more remote. But the least of the Lines that fall upon the convex Circumference*

* 23. 1.
† 4. 1.
* by this.

cumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the least, is less than that which is farther distant; and from that Point there can be drawn only two equal Lines, which shall fall on the Circumference on each Side the least Line; which was to be demonstrated.

PROPOSITION IX.

THEOREM.

If a Point be assumed in a Circle, and from it more than two equal Right Lines be drawn to the Circumference; then that Point is the Centre of the Circle.

LET the Point D be assumed within the Circle ABC; and from the Point D, let there fall more than two equal Right Lines to the Circumference, *viz.* the Right Lines DA, DB, DC. I say, the assumed Point D is the Centre of the Circle ABC.

For, if it be not, let E be the Centre, if possible; and join DE, which produce to G and F.

Then FG is a Diameter of the Circle ABC; and so, because the Point D, not being the Centre of the Circle, is assumed in the Diameter FG; therefore DG
* 7 of this. will * be the greatest Line drawn from D to the Circumference, and DC greater than DB, and DB than DA; but they are also equal, which is absurd. Therefore E is not the Centre of the Circle ABC. And in this manner we prove, *that no other Point, except* D, *is the Centre*; *therefore* D *is the Centre of the Circle* ABC; which was to be demonstrated.

Otherwise:

Let ABC be the Circle, within which take the Point D, from which let more than two equal Right Lines fall on the Circumference of the Circle, *viz.* the three equal ones DA, DB, DC: I say, the Point D is the Centre of the Circle ABC.

† 10. 1. For, join AB, BC; which bisect † in the Points E and Z; as also join ED, DZ; which produce to the

the Points H, K, O, L; then, because AE is equal to EB, and ED is common, the two Sides AE, ED, shall be equal to the two Sides BE, ED. And the Base DA, is equal to the Base DB: Therefore the Angle AED will be * equal to the Angle BED; and so [by *Def.* 10. 1.] each of the Angles AED, BED, is a Right Angle: Therefore HK, bisecting [illegible] cuts it at Right Angles. And because a Right Line in a Circle, bisecting another Right Line, cuts it at Right Angles, and the Centre of the Circle is in the cutting Line, [by *Cor.* 1. 3.] therefore the Centre of the Circle ABC will be in HK. For the same Reason, the Centre of the Circle will be in OL. And the Right Lines HK, OL, have no other Point common but D: *Therefore* D *is the Centre of the Circle* ABC; which was to be demonstrated.

PROPOSITION X.

Theorem.

A Circle cannot cut another Circle in more than two Points.

FOR, if it can, let the Circle ABC cut the Circle DEF in more than two Points, *viz.* in B, G, F; and let K be the Centre of the Circle ABC, and join KB, KG, KF.

Now, because the Point K is assumed within the Circle DEF, from which more than two equal Right Lines KB, KG, KF, fall on the Circumference, the Point K shall be † the Centre of the Circle DEF. † 9 *of this.*
But K is ‡ the Centre of the Circle ABC. Therefore ‡ *By Hyp.*
K will be the Centre of two Circles cutting each other;
which is * absurd. Wherefore, *a Circle cannot cut a* * 5 *of this.*
Circle in more than two Points; which was to be demonstrated.

PROPOSITION XI.

THEOREM.

If two Circles touch each other on the Inside, and the Centres be found, the Line joining their Centres will fall on the [Point of] Contact of those Circles.

LET two Circles ABC, ADE, touch one another inwardly in A; and let F be the Centre of the Circle ABC, and G that of ADE. I say, a Right Line joining the Centres G and F, being produced, will fall in the Point A.

If this be denied, let the Right Line, joining F G, cut the Circles in D and H.

Now, becauſe A G and G F are greater than A F, * that is, than F H; take away F G, which is common, and the Remainder A G is greater than the Remainder G H. But A G is equal to G D; therefore G D is greater than G H, the leſs than the greater; which is abſurd. Wherefore, *a Line drawn thro' the Points* F *and* G, *will not fall out of the Point of Contact* A, *and ſo neceſſarily muſt fall on it*; which was to be demonſtrated.

PROPOSITION XII.

THEOREM.

If two Circle touch one another on the Outſide, a Right Line joining their Centres will paſs thro' the [Point of] Contact.

LET two Circles ABC, ADE, touch one another outwardly in the Point A; and let F be the Centre of the Circle ABC, and G that of ADE. I ſay, a Right Line drawn thro' the Centres F and G, will paſs thro' the Point of Contact A.

For, if it does not, let, if poſſible, FCDG, fall without it, and join F A, A G.

Now, ſince F is the Centre of the Circle ABC, AF will be equal to F C. And becauſe G is the Centre of the

the Circle ADE, AG will be equal to GD: But AF has been shewn to be equal to FC; therefore FA and AG are equal to FC and DG. And so the whole FG is greater than FA and AG; and also less, * which is absurd. Therefore, *a Right Line, drawn from the Point* F *to* G, *will pass thro' the Point of Contact* A; which was to be demonstrated. * 20. 1.

ITION XIII.

THEOREM.

One Circle cannot touch another in more Points than one, whether it be inwardly or outwardly.

FOR, in the first Place, if this be denied, let the Circle ABDC, if possible, touch the Circle EBFD inwardly, in more Points than one, *viz.* in B, and D.

And let G be the Centre of the Circle ABDC, and H that of EBFD.

Then a Right Line, drawn from the Point G to H, will † fall in the Points B and D. Let this Line be BGHD. And because G is the Centre of the Circle ABDC, the Line BG will be equal to GD. Therefore BG is greater than HD, and BH much greater than HD. Again, since H is the Centre of the Circle EBFD, the Line BH is equal to HD. But it has been proved to be much greater than it, which is absurd. Therefore, *one Circle cannot touch another Circle inwardly in more Points than one.* † 11 of this.

Secondly, Let the Circle ACK, if possible, touch the Circle ABDC outwardly, in more Points than one, *viz.* in A and C; and let A and C be joined.

Now, because two Points, A and C, are assumed in the Circumference of each of the Circles ABDC, ACK, a Right Line joining these two Points will fall ‡ within either of the Circles. But it falls within the Circle ABDC, and without the Circle ACK, which is absurd. Therefore one Circle cannot touch another Circle in more Points than one outwardly. But it has been proved, that one Circle cannot touch another Circle inwardly [in more Points than one]. Wherefore, *one Circle cannot touch another in more Points than one, whether* ‡ 2 of this.

ther it be inwardly or outwardly; which was to be demonstrated.

PROPOSITION XIV.

THEOREM

Equal Right Lines in a Circle are equally distant from the Centre; and Right Lines, which are equally distant from the Centre, are equal between themselves.

LET ABDC be a Circle, wherein are the equal Right Lines AB, CD. I say, these Lines are equally distant from the Centre of the Circle.

For, let E be the Centre of the Circle ABDC; from which let there be drawn EF and EG, perpendicular to AB and CD; and let AE and EC be joined.

Then, because a Right Line EF, drawn thro' the Centre, cuts the Right Line AB, not drawn thro' the
3 *of this.* Centre, at Right Angles, it will * bisect the same. Wherefore AF is equal to FB, and so AB is double to AF. For the same Reason CD is double to CG; but AB is equal to CD; therefore AF is equal to CG: And because AE is equal to EC, the Square of AE shall be equal to the Square of EC. But the
† 47. 1. Squares of AF and FE are † equal to the Square of AE; for the Angle at F is a Right Angle: And the Squares of EG and GC are equal to the Square of EC, since the Angle at G is a Right one. Therefore the Squares of AF and FE are equal to the Squares of CG and GE. But the Square of AF is equal to the Square of CG; for AF is equal to CG. Therefore the Square of FE is equal to the Square of EG; and
‡ *Def.* 4. *of this.* so FE equal to EG. Also Lines in a Circle are ‡ said to be equally distant from the Centre, when Perpendiculars drawn to them from the Centre are equal. Therefore, AB *and* CD *are equally distant from the Centre.*

But if AB and CD are equally distant from the Centre, that is, if FE be equal to EG, I say, AB is equal to CD.

For, the same Construction being supposed, we demonstrate, as above, that AB is double to AF, and

CD

CD to CG; and because AE is equal to EC, the Square of AE will be equal to the Square of EC. But the Squares of EF and FA are † equal to the Square of AE; also the Squares of EG and GC are equal † to the Square of EC. Therefore the Squares of EF and FA are equal to the Squares of EG and GC. But the Square of EG is equal to the Square of EF; for EG is equal to EF. Therefore the Square of AF is equal to the Square of CG; and so AF is equal to CG. But AB is double to AF, and CD to CG; whence AB is equal to CD. Therefore, *equal Right Lines in a Circle are equally distant from the Centre; and Right Lines, which are equally distant from the Centre, are equal between themselves*; which was to be demonstrated. † 47. 1.

PROPOSITION XV.

Theorem.

A Diameter is the greatest Line in a Circle; and of all the other Lines therein, that which is nearest to the Centre is greater than that more remote.

LET ABCD be a Circle whose Diameter is AD, and Centre E; and let BC be nearer to the Centre than FG. I say, AD is the greatest, and BC is greater than FG.

For, let the Perpendiculars EH, EK, be drawn from the Centre E to BC, FG. Now, because BC is nearest to the Centre than FG, EK will be greater than EH. Let EL be equal to EH; draw LM thro' L at Right Angles to EK, which produce to N; and let EM, EN, EF, EG, be joined.

Then, because EH is equal to EL, the Line BC will be equal to MN *. And, since AE is equal to EM, and DE to EN, AD will be equal to ME and EN. But ME and EN are † greater than MN: And so AD is greater than MN; and NM is equal to BC. Therefore AD is greater than BC. And since the two Sides EM, EN, are equal to the two Sides FE, EG, and the Angle MEN greater than the Angle FEG, the Base MN shall be ‡ greater than the Base FG. But MN is equal to BC. Therefore BC is greater than FG.

* 14 of this.
† 20. 1.
‡ 24. 1.

F G. And so the Diameter A D is the greatest, and B C is greater than F G. Wherefore, *the Diameter is the greatest Line in a Circle; and of all the other Lines therein, that which is nearest to the Centre is greater than that more remote*; which was to be demonstrated.

PROPOSITION XVI.

THEOREM.

A Line drawn from the extreme [Point] of the Diameter of a Circle, at Right Angles to that Diameter, shall fall without the Circle; and between the said Right Line, and the Circumference, no other Right Line can be drawn; and the Angle of a Semicircle is greater than any Right-lined acute Angle; and the remaining Angle [viz. *without the Circumference*] *is less than any Right-lined Angle.*

LET ABC be a Circle, whose Centre is D, and Diameter A B. I say, a Right Line, drawn from the Point A at Right Angles to A B, falls without the Circle.

For, if it does not, let it fall, if possible, within the Circle, as AC; and join DC.

Now, because DA is equal to DC, the Angle DAC shall be * equal to the Angle ACD. But DAC is a Right Angle; therefore ACD is a Right Angle: And accordingly the Angles DAC, ACD, are equal to two

† 17. 1. Right Angles; which is absurd †. Therefore a Right Line, drawn from the Point A at Right Angles to BA, will not fall within the Circle; and so likewise we prove, that it neither falls in the Circumference. Therefore, *it will necessarily fall without the same*; which now let be AE.

Again, between the Right Line A E and the Circumference CHA, no other Right Line can be drawn.

‡ 12. 1. For, if there can, let it be FA, and let ‡ DG be drawn, from the Centre D, at Right Angles to FA.

Now, because AGD is a Right Angle, and DAG is less than a Right Angle, DA will be greater than

19. 1. DG *. But DA is equal to DH. Therefore DH is

greater

greater than DG, the less than the greater; which is absurd. Wherefore, *no Right Line can be drawn between* AE, *and the Circumference* AHC. I say, moreover, that the Angle of the Semicircle, contained under the Right Line BA, and the Circumference CHA, is greater than any Right-lined acute Angle; and the remaining Angle contained under the Circumference CHA, and the Right Line AE, is less than any Right-lined Angle.

For if any Right-lined acute Angle be greater than the Angle contained under the Right Line BA, and the Circumference CHA; or if any Right-lined Angle be less than that contained under the Circumference CHA, and the Right Line AE; then a Right Line may be drawn between the Circumference CHA, and the Right Line AE, making an Angle (contained under Right Lines) greater than that contained under the Right Line BA, and the Circumference CHA, and less than that contained under the Circumference CHA, and the Right Line AE. But such a Right Line cannot be drawn, from what has been proved. Therefore, *no Right-lined acute Angle is greater than the Angle contained under the Right Line* BA, *and the Circumference* CHA; *nor less than the Angle contained under the Circumference* CHA, *and the Right Line* AE; which was to be demonstrated.

Coroll. From hence it is manifest, that a Right Line, drawn at Right Angles, on the End of the Diameter, of a Circle, touches the Circle, and that in one Point only, because, if it should meet it in two Points, it would fall within the same; * *as has been demonstrated.* * 2 *of this.*

PROPOSITION XVII.

PROBLEM.

To draw a Right Line from a given Point, that shall touch a given Circle.

LET A be the Point given, and BCD the Circle. It is required to draw a Right Line from the Point A, that shall touch the given Circle BCD.

Let E be the Centre of the Circle, and join A E; then about the Centre E, with the Distance EA, describe the Circle A F G; draw D F * at Right Angles to E A, and join E B F, and A B. I say, the Right Line A B is drawn from the Point A, touching the Circle B C D.

* 11. 1.

For, since E is the Centre of the Circles B C D, AFG, the Line EA will be equal to EF, and ED to E B. Therefore the two Sides A E, E B, are equal to the two Sides F E, E D, each to each; and they contain the common Angle E. Wherefore the Base DF is † equal to the Base A B, and the Triangle D E F equal to the Triangle E B A, and the remaining Angles of the one equal to the remaining Angles of the other. And so the Angle E B A is equal to the Angle EDF. But EDF is a Right Angle. Wherefore EBA is also a Right Angle, and E B is a Line drawn from the Centre; but a Right Line, drawn from the Extremity of the Diameter of a Circle at Right Angles ‡ to it, touches the Circle. Wherefore, AB *touches the Circle*; which was to be done.

† 4. 1.

‡ Cor. 16. of this.

PROPOSITION. XVIII.

THEOREM.

If any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent.

LET any Right Line D E touch a Circle A B C in the Point C, and let there be drawn the Right Line F C from the Centre F. I say, F C is perpendicular to D E.

For, if it be not, let FG be drawn * from the Centre F, perpendicular to D E.

* 12. 1.

Now, because the Angle F G C is a Right Angle, the Angle G C F will be † an acute Angle; and accordingly the Angle F G C is greater than the Angle F C G; but the greater Angle subtends ‡ the greater Side. Therefore F C is greater than F G. But F C is equal to FB. Wherefore FB is greater than FG,

† Cor. of 32. 1.

‡ 19.

a less than a greater; which is absurd. Therefore FG is not perpendicular to DE. And in the same manner we prove, that no other Right Line but FC is perpendicular to DE. Wherefore FC is perpendicular to DE. Therefore, *if any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn, that Line will be perpendicular to the Tangent*; which was to be demonstrated.

PROPOSITION XIX.

THEOREM.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line.

LET any Right Line DE touch the Circle ABC in C, and let CA be drawn from the Point C at Right Angles to DE. I say, the Circle's Centre is in AC.

For if it be not, let F be the Centre, if possible; and join CF.

Then, because the Right Line DE touches the Circle ABC, and FC is drawn from the Centre to the Point of Contact; FC will be perpendicular to DE *. And so the Angle FCE is a Right one. But ACE is also a Right Angle †. Therefore the Angle FCE is equal to the Angle [redacted] a less to a greater; which is absurd. Therefore F is not the Centre of the Circle ABC. After this manner we prove, that the Centre of the Circle can be in no other Line, but AC. Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle shall be in the said Line*; which was to be demonstrated.

* 18 *of this.*

† *From the Hyp.*

ACE

PROPOSITION XX.

THEOREM.

The Angle at the Centre of the Circle is double to the Angle at the Circumference, when the same Arc is the Base of both Angles.

LET ABC be a Circle, at the Centre E whereof is the Angle BEC, and at the Circumference, the Angle BAC, both of which ſtand upon the ſame Arc BC. I ſay, the Angle BEC is double to the Angle BAC.

For join A and produce it to F.

Then, becauſe EA is equal to EB, the Angle EAB
* 5. 1. ſhall be equal to the Angle EBA *. Therefore the Angles EAB, EBA, are double to the Angle EAB;
† 32. 1. but the Angle BEF is † equal to the Angles EAB, EBA; therefore the Angle BEF is double to the Angle EBA. For the ſame Reaſon, the Angle FEC is double to EAC. Therefore the whole Angle BEC is double to the whole Angle BAC. Again, let there be another Angle BDC; and join DE, which produce to G. We demonſtrate, in the ſame manner, that the Angle GEC is double to the Angle GDC; whereof the Part GEB is double to the Part GDB. And therefore the remaining Part BEC is double to the remaining Part BDC. Conſequently, *an Angle at the Centre of a Circle is double to the Angle at the Circumference, when the ſame Arc is the Baſe of both Angles*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

Angles that are in the ſame Segment of a Circle, are equal to each other.

LET ABCDE be a Circle, and let BAD, BED, be Angles in the ſame Segment thereof BAED. I ſay, thoſe Angles are equal.

For,

For, let F be the Centre of the Circle ABCDE; and join BF, FD.

Now, because the Angle BFD is at the Centre, and the Angle BAD at the Circumference, and they stand upon the same Arc BCD; the Angle BFD will be ‡ ‡ 20 *of this.*
double to the Angle BAD. For the same Reason, the Angle BFD is also double to the Angle BED. Therefore the Angle BAD will be equal to the Angle BED.

If the Angles BAD, BED, are in a Segment less than a Semicircle, let AE be drawn; and then all the Angles of the Triangle ABG are † equal to all the † 32. 1.
Angles of the Triangle DEG. But the Angles ABE, ADE, are equal, from what has been before proved; and the Angles AGB, DGE, are also equal ‡; for ‡ 15. 1.
they are vertical Angles. Wherefore the remaining Angle BAG is equal to the remaining Angle GED. Therefore, *Angles that are in the same Segment of a Circle, are equal to each other*; which was to be demonstrated.

PROPOSITION XXII.

Theorem.

The opposite Angles of any quadrilateral Figure, described in a Circle, are equal to two Right Angles.

LET ABDC be a Circle, wherein is described the quadrilateral Figure ABCD. I say, two opposite Angles thereof are equal to two Right Angles.

For join AD, BC.

Then, because the three Angles of any Triangle are * equal to two Right Angles, the three Angles of the * 32. 1.
Triangle ABC, *viz.* the Angles CAB, ABC, BCA, are equal to two Right Angles. But the Angle ABC is † equal to the Angle ADC; for they are both in † 21 *of this.*
the same Segment ABDC. And the Angle ACB is † equal to the Angle ADB, because they are in the same Segment ACDB; therefore the whole Angle BDC is equal to the Angles ABC, ACB; and if the common Angle BAC be added, then the Angles BAC, ABC, ACB, are equal to the Angles BAC, BDC; but the Angles BAC, ABC, ACB, are equal ‡ to two Right ‡ 32. 1.

 Angles,

Angles. Therefore likewife, the Angles BAC, BDC, fhall be equal to two Right Angles. And after the fame Way we prove, that the Angles ABD, ACD, are alfo equal to two Right Angles. Therefore, *the oppofite Angles of any quadrilateral Figure, defcribed in a Circle, are equal to two Right Angles*; which was to be demonftrated.

PROPOSITION XXIII.

THEOREM.

Two fimilar and unequal Segments of two Circles cannot be fet upon the fame Right Line, and on the fame Side thereof.

FOR if this be poffible, let the two fimilar and unequal Segments ACB, ADB, of two Circles, ftand upon the Right Line AB on the fame Side thereof. Draw ACD, and let CB, BD, be joined. Now, becaufe the Segment ACB is fimilar to the Segment
* *Def.* 11. *of this.* ADB, and fimilar Segments of Circles are * fuch which include equal Angles; the Angle ACB will be equal to the Angle ADB; the outward one to the in-
† 16. 1. ward one; which is † abfurd. Therefore, *fimilar and unequal Segments of two Circles cannot be fet upon the fame Right Line, and on the fame Side thereof*; which was to be demonftrated.

PROPOSITION XXIV.

THEOREM.

Similar Segments of Circles, being upon equal Right Lines, are equal to one another.

LET AEB, CFD, be fimilar Segments of Circles, ftanding upon the equal Right Lines AB, CD. I fay, the Segment AEB is equal to the Segment CFD.

For the Segment AEB being applied to the Segment CFD, fo that the Point A coincides with C, and the Line AB with CD; then the Point B will coincide with the Point D, fince AB and CD are equal,

equal. And since the Right Line AB coincides with CD, the Segment AEB will coincide with the Segment CFD. For if, at the same Time that AB coincides with CD, the Segment AEB should not coincide with the Segment CFD, but be otherwise, as CGD; then a Circle would cut a Circle in more Points than two, *viz.* in the Points C, G, D; which is * impossible. Wherefore, if the Right Line AB coincides with CD, the Segment AEB will coincide with, and be equal to, the Segment CFD. Therefore, *similar Segments of Circles, being upon equal Right Lines, are equal to one another*; which was to be demonstrated. * 10 of this.

PROPOSITION XXV.

PROBLEM.

A Segment of a Circle being given, to describe the Circle whereof it is the Segment.

LET ABC be a Segment of a Circle given. It is required to describe a Circle, whereof ABC is a Segment.

Bisect * AC in D, and let DB be drawn † from the Point D at Right Angles to AC; and join AB. Now the Angle ABD is either greater, equal, or less, than the Angle BAD. And first let it be greater and make ‡ the Angle BAE at the given Point A, with the Right Line BA, equal to the Angle ABD; produce BD to E, and join EC. * 10. 1. † 11. 1. ‡ 23. 1.

Then, because the Angle ABE is equal to the Angle BAE, the Right Line BE will be * equal to EA. And because AD is equal to DC, and DE common, the two Sides AD, DE, are each equal to the two Sides CD, DE; and the Angle ADE is equal to the Angle CDE; for each is a Right one. Therefore the Base AE is equal to the Base EC. But AE has been proved to be equal to EB. Wherefore BE is also equal to EC. And accordingly the three Right Lines AE, EB, EC, are equal to each other. Therefore a Circle described about the Centre E, with either of the Distances AE, EB, EC, † shall pass thro' the other Points, and be that required to be described. But it is manifest, that the Segment ABC is less than a Semi- * 6. 1. † 9 of this.

a Semicircle, because the Centre thereof is without the same.

But if the Angle ABD be equal to the Angle BAD; then if AD be made equal to BD, or DC, the three Right Lines AD, BD, DC, are equal between themselves, and D will be the Centre of the Circle to be described, and the Segment ABC is a Semicircle.

But if the Angle ABD is less than the Angle BAD, let the Angle BAE be made, at the given Point A, with the Right Line BA, within the Segment ABC, equal to the Angle ABD.

Then the Point E, in the Right Line DB, will, by arguing as before, appear to be the Centre, and ABC a Segment greater than a Semicircle. Therefore, *a Circle is described, whereof a Segment is given*; which was to be done.

PROPOSITION XXVI.

THEOREM.

In equal Circles, equal Angles stand upon equal Circumferences, whether they be at their Centres, or at their Circumferences.

LET ABC, DEF, be equal Circles; and let BGC, EHF, be equal Angles at their Centres; and BAC, EDF, equal Angles at their Circumferences. I say, the Circumference BKC is equal to the Circumference ELF.

For, let BC, EF, be joined. Because ABC, DEF, are equal Circles, the Lines drawn from their Centres
‡ Def. 1. will ‡ be equal. Therefore the two Sides BG, GC, are equal to the two Sides EH, HF; and the Angle G is equal to the Angle H. Wherefore the Base
* 4. 1. BC is * equal to the Base EF. Again, because the Angle at A is equal to that at D, the Segment BAC
† Def. 11. will be † similar to the Segment EDF; and they are upon equal Right Lines BC, EF. But those similar Segments of Circles, that are upon equal Right Lines,
‡ 24 of this. are ‡ equal to each other. Therefore the Circumfe-
† Def. 11. rence BAC will be † equal to the Circumference EDF. But the whole Circumference ABCA is equal to the whole Circumference DEFD. Therefore the remain-

ing Circumference BKC shall be equal to the remaining Circumference ELF. Therefore, *in equal Circles, equal Angles stand upon equal Circumferences, whether they be at their Centres, or at their Circumferences*; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

Angles, that stand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres, or Circumferences.

LET the Angles BGC, EHF, at the Centres of the equal Circles ABC, DEF, and the Angles BAC, EDF, at their Circumferences, stand upon the equal Circumferences BC, EF. I say, the Angle BGC is equal to the Angle EHF, and the Angle BAC to the Angle EDF.

For if the Angle BGC be equal to the Angle EHF, it is manifest, that the Angle BAC is also equal to the Angle EDF: But if the Angle BGC be not equal to the Angle EHF, let one of them be the greater, as BGC, and make * the Angle BGK, at the Point G, with the Line BG, equal to the Angle EHF. But equal Angles stand † upon equal Circumferences, when they are at the Centres. Wherefore the Circumference BK is equal to the Circumference EF. But the Circumference EF is equal to the Circumference BC. Therefore BK is equal to BC, a less to a greater, which is absurd. Wherefore the Angle BGC is not unequal to the Angle EHF, and so it must be equal to it. But the Angle at A is one half of the Angle BGC; and the Angle at D is one half of the Angle EHF. Therefore the Angle at A is equal to the Angle at D. Wherefore, *Angles, that stand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres, or Circumferences*; which was to be demonstrated.

* 23. 1.

† 26 *of this.*

PRO-

PROPOSITION XXVIII.

THEOREM.

In equal Circles, equal Right Lines cut off equal Parts of the Circumferences; the greater Part of the one Circumference equal to the greater Part of the other, and the lesser, equal to the lesser.

LET ABC, DEF, be equal Circles, in which are the equal Right Lines BC, EF, which cut off the greater Circumferences BAC, EDF, and the lesser Circumferences BGC, EHF. I say, the greater Circumference BAC is equal to the greater Circumference EDF, and the lesser Circumference BGC to the lesser Circumference EHF.

For assume the Centres K and L of the Circles; and join BK, KC, EL, LF.

Because the Circles are equal, the Lines drawn
* *Def.* 1. from their Centres * are also equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and the Base BC is equal to the Base EF.
† 8. 1. Therefore the Angle BKC is † equal to the Angle
‡ 26 *of this.* ELF. But equal Angles stand ‡ upon equal Circumferences, when they are at the Centres. Wherefore the Circumference BGC is equal to the Circumference EHF, and the whole Circumference ABCA equal to the whole Circumference DEFD; and so the remaining Circumference BAC shall be equal to the remaining Circumference EDF. Therefore, *in equal Circles, equal Right Lines cut off equal Parts of the Circumferences*; which was to be demonstrated.

PROPOSITION XXIX.

THEOREM.

In equal Circles, the Right Lines, which subtend equal Circumferences, are equal.

LET there be two equal Circles, ABC, DEF; and let the equal Circumferences BGC, EHF, be assumed in them, and BC, EF, joined. I say, the

the Right Line BC is equal to the Right Line EF.

For, find * K and L, the Centres of the Circles; and join BK, KC, EL, LF. * 1 of this.

Then, becauſe the Circumference BGC is equal to the Circumference EHF, the Angle BKC ſhall be † equal to the Angle ELF. And becauſe the Circles ABC, DEF, are equal, the Lines drawn from their Centres ſhall be ‡ equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and they contain equal Angles: Wherefore the Baſe BC is † equal to the Baſe EF. And ſo, *in equal Circles, the Right Lines, which ſubtend equal Circumferences, are equal*; which was to be demonſtrated. † 27 of this. ‡ Def. 1. † 4. 1.

PROPOSITION XXX.

PROBLEM.

To cut a given Circumference into two equal Parts.

LET the given Circumference be ADB. It is required to cut the ſame into two equal Parts.

Join AB, which biſect * in C; and let the Right Line CD be drawn from the Point C at Right Angles to AB †; and join AD, DB. * 10. 1. † 11. 1.

Now, becauſe AC is equal to CB, and CD is common, the two Sides AC, CD, are equal to the two Sides BC, CD; but the Angle ACD is equal to the Angle BCD; for each of them is a Right Angle: Therefore the Baſe AD is † equal to the Baſe BD. But equal Right Lines cut ‡ off equal Circumferences. Wherefore the Circumference AD ſhall be equal to the Circumference BD. Therefore, *a given Circumference is cut into two equal Parts*; which was to be done. † 4. 1. ‡ 28 of this.

PRO-

PROPOSITION XXXI.

Theorem.

In a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment is less than a Right Angle; and the Angle in a lesser Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a lesser Segment is less than a Right Angle.

LET there be a Circle AB[illegible]hose Diameter is BC, and Centre E; and joi[illegible], AC, AD, DC. I say, the Angle which is in the Semicircle BAC is a Right Angle; that which is in the Segment ABC being greater than a Semicircle, *viz.* the Angle ABC is less than a Right Angle; and that which is in the Segment ADC being less than a Semicircle; that is, the Angle ADC is greater than [illegible]ght Angle.

For, join AE, and produ[illegible]A to F.

Then, because BE is equal to EA, the Angle EAB shall be * equal to the Angle EBA. And because AE is equal to EC, the Angle ACE will be * equal to the Angle CAE. Therefore the whole Angle BAC, is equal to the two Angles ABC, ACB; but the Angle FAC, being without the Triangle ABC, is † equal to the two Angles ABC, ACB; therefore the Angle BAC is equal to the Angle FAC; and so each of them is ‡ a Right Angle. Wherefore, *the Angle* BAC, *in a Semicircle, is a Right Angle.* And because the two Angles ABC, BAC, of the Triangle ABC *, are less than two Right Angles, and BAC is a Right Angle; then, ABC *is less than a Right Angle; and is, in the Segment* ABC, *greater than a Semicircle.*

* 5. 1.
† 32. 1.
‡ Def. 10. 1.
* 17. 1.

And since ABCD is a quadrilateral Figure in a Circle, and the opposite Angles of any quadrilateral Figure described in a Circle are † equal to two Right Angles; the Angles ABC, ADC, are equal to two Right Angles; and the Angle ABC is less than a Right Angle. Therefore, *the remaining Angle* ADC

† 22 of this.

will

will be greater than a Right Angle; and is in the Segment ADC, *which is less than a Semicircle.*

I say, moreover, the Angle of the greater Segment contained under the Circumference ABC and the Right Line AC, is greater than a Right Angle; and the Angle of the lesser Segment, contained under the Circumference ADC, and the Right Line AC, is less than a Right Angle. This manifestly appears; for, because the Angle contained under the Right Lines BA, AC, is a Right Angle; the Angle contained under the Circumference ABC, and the Right Line AC, will be greater than a Right Angle. Again, because the Angle contained under the Right Lines CA, AF, is a Right Angle, therefore the Angle which is contained under the Right Line AC, and the Circumference ADC, is less than a Right Angle. Therefore, *in a Circle, the Angle that is in the Semicircle is a Right Angle; but the Angle in a greater Segment is less than a Right Angle; and the Angle in a lesser Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a lesser Segment is less than a Right Angle;* which was to be demonstrated.

PROPOSITION XXXII.

Theorem.

If any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle.

LET any Right Line EF touch the Circle ABCD in the Point B, and let the Right Line BD be any how drawn from the Point B, cutting the Circle. I say, the Angles which BD makes with the Tangent Line EF, are equal to those in the alternate Segments of the Circle; that is, the Angle FBD is equal to an Angle made in the Segment DAB, *viz.* to the Angle DAB; and the Angle DBE equal to the Angle DCB, made in the Segment DCB. For,

Draw

* 11. 1. Draw * BA, from the Point B, at Right Angles to EF; and take any Point C, in the Circumference BD; and join AD, DC, CB.

Then, because the Right Line EF touches the Circle ABCD in the Point B; and the Right Line BA is drawn from the Point of Contact B, at Right Angles to the Tangent Line; the Centre of the Circle † 19 *of this.* ABCD will † be in the Right Line BA; and so BA is a Diameter of the Circle, and the Angle ADB, ‡ 31 *of this.* in a Semicircle is ‡ a Right Angle. Therefore the * 32. 1. other Angles, BAD, ABD, are * equal to one Right Angle. But the Angle ABF is also a Right Angle: Therefore the Angle ABF is equal to the Angles BAD, ABD; and if ABD, [redacted]ich is common, be [redacted] taken away, then the Angle [redacted], remaining, will be equal to that which is the alternate Segment of the Circle, *viz.* equal to the Angle BAD. And because ABCD is a quadrilateral Figure in a Circle, the † 22 *of this.* opposite Angles thereof are † equal to two Right Angles; therefore the Angles DBF, DBE, will be equal to the Angles BAD, BCD. But BAD has [redacted] been proved to be equal to [redacted]; therefore the Angle DBE is equal to the Angle made in DCB, the alternate Segment of the Circle, *viz.* equal to the Angle DCB. Therefore, *if any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle*; which was to be demonstrated.

PROPOSITION XXXIII.

PROLBEM.

To describe upon a given Right Line, a Segment of a Circle, which shall contain an Angle, equal to a given Right-lined Angle.

LET the given Right Line be AB, and C the given Right-lined Angle. It is required to describe the Segment of a Circle upon the given Right Line AB, containing an Angle, equal to the Angle C.

At

At the Point A, with the Right Line AB, make ‡ the Angle BAD equal to tho Angle C, and draw ‡ 23. 1.
* AE from the Point A, at Right Angles to AD. * 11. 1.
Likewise bisect † AB in F, and let FG be drawn † 10. 1.
from the Point F, at Right Angles to AB; and join GB.

Then, because AF is equal to FB, and FG is common, the two Sides AF, FG, are equal to the two Sides BF, FG; and the Angle AFG is equal to the Angle BFG. Therefore the Base AG is ‡ equal to ‡ 4. 1.
the Base GB. And so, if a Circle be described about the Centre G, with the Distance A G, this shall pass thro' the Point B. Describe the Circle, which let be A B E, and join E B. Now, because A D is drawn from the Point A[redacted] Extremity of the Diameter [redacted]
A E, at Right Angles to A E, the said A D will * * *Cor.* 16. *of*
touch the Circle. And since the Right Line A D *this.*
touches the Circle A B E, and the Right Line A B is drawn in the Circle from the Point of Contact A, the Angle D A B is † equal to the Angle made in the alter- † 32 *of this.*
nate Segment, *viz.* equal to the Angle A E B. But the Angle D A B is [redacted] to the Angle C. Therefore [redacted]
the Angle C will be equal to the Angle AEB. Wherefore, *the Segment of a Circle* A E B *is described upon the given Right Line* A B, *containing an Angle* A E B, *equal to a given Angle* C; which was to be done.

PROPOSITION XXXIV.

PROLBEM.

To cut off a Segment from a given Circle, that shall contain an Angle, equal to a given Right-lined Angle.

LET the given Circle be A B C, and the Right-lined Angle given D. It is required to cut off a Segment from the Circle A B C, containing an Angle equal to the Angle D.

Draw ‡ the Right Line E F, touching the Circle in ‡ 17 *of this.*
the Point B, and make * the Angle FBC, at the Point * 23. 1.
B, equal to the Angle D.

Then, because the Right Line E F touches the Circle A B C in the Point B, and BC is drawn from the

* 32 *of this.* the Point of Contact B; the Angle FBC will be * equal to that in the alternate Segment of the Circle; but the Angle FBC is equal to the Angle D. Therefore the Angle in the Segment BAC will be equal to the Angle D. Therefore, *the Segment* BAC *is cut off from the given Circle* ABC, *containing an Angle equal to the given Right-lined Angle* D; which was to be done.

PROPOSITION XXXV.

THEOREM.

If two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one is equal to the Rectangle under the Segments of the other.

IN the Circle ABCD, let two Right Lines mutually cut each other in the Point E. I say, the Rectangle contained under AE and EC is equal to the Rectangle contained under DE and EB.

If AC and DB pass through the Centre, so that E be the Centre of the Circle ABCD; it is manifest, since AE, EC, DE, EB, are equal, that the Rectangle under AE and EC is equal to the Rectangle under DE and EB.

But if AC, DB, do not pass through the Centre, assume the Centre of the Circle F; from which draw FG, FH, perpendicular to the Right Lines AC, DB; and join FB, FC, FE.

Then, because the Right Line GF, drawn through the Centre, cuts the Right Line AC, not drawn thro' * 3 *of this.* the Centre, at Right Angles, it will also bisect * the same. Wherefore AG is equal to GC: And because the Right Line AC is cut into two equal Parts in the Point G, and into two unequal Parts in E, the Rectangle under AE and EC, together with the Square † 5. 2. of EG, is † equal to the Square of GC. And if the common Square of GF be added, then the Rectangle under AE and EC, together with the Squares of EG and GF, is equal to the Squares of CG and GF. ‡ 47. 1. But the Square of FE is ‡ equal to the Squares of EG and GF, and the Square of FC equal ‡ to the Squares

of CG and GF. Therefore the Rectangle under AE and EC, together with the Square of FE, is equal to the Square of FC; but CF is equal to FB. Therefore the Rectangle under AE and EC, together with the Square of EF, is equal to the Square of FB. For the same Reason, the Rectangle under DE and EB, together with the Square of FE, is equal to the Square of FB. But it has been proved, that the Rectangle under AE and EC, together with the Square of FE, is also equal to the Square of FB. Therefore the Rectangle under AE and EC, together with the Square of FE, is equal to the Rectangle under DE and EB, together with the Square of FE. And if the common Square of FE be taken away, then there will remain the Rectangle under AE and EC, equal to the Rectangle under DE and EB. Wherefore, *if two Right Lines in a Circle mutually cut each other, the Rectangle, contained under the Segments of the one, is equal to the Rectangle, under the Segments of the other*; which was to be demonstrated.

PROPOSITION XXXVI.

Theorem.

If some Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its Part between the Convexity of the Circle and the assumed Point, will be equal to the Square of the Tangent Line.

LET any Point D be assumed without the Circle ABC, and let two Right Lines DCA, DB, fall from the said Point to the Circle; whereof DCA cuts the Circle, and DB touches it. I say, the Rectangle under AD and DC is equal to the Square of DB.

Now DCA either passes thro' the Centre, or not. In the first Place, let it pass thro' the Centre of the Circle ABC, which let be E, and join EB. Then the Angle EBD is * a Right Angle. And so, since the Right Line AC is bisected in E, and CD is added thereto, the Rectangle under AD and DC, together * 18 *of this.*

 with

6. 2. with the Square of EC, shall * be equal to the Square of ED. But EC is equal to EB; wherefore the Rectangle under AD and DC, together with the Square of EB, is equal to the Square of ED. But the Square
† 47. 1. of ED is † equal to the Squares of EB and BD, for the Angle EBD is a Right Angle: Therefore the Rectangle under AD and DC, together with the Square of EB, is equal to the Squares of EB and BD; and if the common Square of EB be taken away, the Rectangle under AD and DC remaining, will be equal to the Square of the Tangent Line BD.

Now, let DCA not pass thro' the Centre of the
‡ 1 *of this.* Circle ABC; and find ‡ the Centre E thereof. and draw EF perpendicular to AC, and join EB, EC, ED. Therefore EFD is a Right Angle. And because a Right Line EF, drawn thro' the Centre, cuts a Right Line AC, not drawn thro' the Centre, at Right
* 3 *of this.* Angles, it will * bisect the same; and so AF is equal to FC. Again, since the Right Line AC is bisected in F, and CD is added thereto, the Rectangle under AD and DC, together with the Square of FC, will
* 6. 2. be * equal to the Square of FD. And if the common Square of EF be added, then the Rectangle under AD and DC, together with the Squares of FC and FE, is equal to the Squares of DF and FE. But the Square of DE is equal to the Squares of DF and FE; for the Angle EFD is a Right one; and the
† 47. 1. Square of CE is † equal to the Squares of CF and FE. Therefore the Rectangle under AD and DC, together with the Square of CE, is equal to the Square of ED; but CE is equal to EB. Wherefore the Rectangle under AD and DC, together with the Square of EB, is equal to the Square of ED. But the Squares of EB and BD are † equal to the Square of ED; since the Angle EBD is a Right one. Wherefore the Rectangle under AD and DC, together with the Square of EB, is equal to the Squares of EB and BD. And if the common Square of EB be taken away, the Rectangle under AD and DC, remaining, will be equal to the Square of DB. Therefore, *if any Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its Part between the Convexity of the*

Circle and the assumed Part, will be equal to the Square of the Tangent Line; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If some Point be taken without a Circle, and two Right Lines be drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this last Line will touch the Circle.

LET some Point D be assumed without the Circle ABC, and from it draw two Right Lines DCA, DB, to the Circle, in such manner, that DCA cuts the Circle, and DB falls upon it: And let the Rectangle under AD and DC be equal to the Square of DB. I say, the Right Line DB touches the Circle.

For, let the Right Line DE be drawn * touching the Circle ABC, and find F the Centre ‡ of the Circle; and join EF, FB, FD. * 17 *of this.* ‡ 1 *of this.*

Then the Angle FED is † a Right Angle. And because DE touches the Circle ABC, and DCA cuts it, the Rectangle under AD and DC will be equal to the Square of DE. But the Rectangle under AD and DC is ‡ equal to the Square of DB. Wherefore the Square of DE shall be equal to the Square of DB. And so the Line DE will be equal to the Line DB. But EF is equal to FB: Therefore the two Sides DE, EF, are equal to the two Sides DB, BF; and the Base FD is common. Wherefore the Angle DEF is equal * to the Angle DBF: But DEF is a Right Angle; wherefore DBF is also a Right Angle, and FB produced is a Diameter. But a Right Line drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle; therefore BD necessarily touches the Circle. We prove this in the same manner, if the Centre of the Circle be in the Right Line CA. *If therefore, any Point be assumed without a Circle, and two Right Lines be* † 18 *of this.* ‡ *By Hyp.* * 8. 1.

drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this last Line will touch the Circle; which was to be demonstrated.

Coroll. Hence, if from any Point, without a Circle, several Right Lines AB, AC, are drawn, cutting the Circle, the Rectangles comprehended under the whole Lines AB, AC, and their external Parts AE, AF, are equal between themselves. For, if the Tangent AD be drawn, the Rectangle under BA and AE is equal to the Square of AD; and the Rectangle under CA and AF is equal to the same Square of AD: Therefore the Rectangles shall be equal.

The END *of the* THIRD BOOK.

drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this last Line will touch the Circle; which was to be demonstrated.

Coroll. Hence, if from any Point, without a Circle, several Right Lines AB, AC, are drawn, cutting the Circle, the Rectangles comprehended under the whole Lines AB, AC, and their external Parts AE, AF, are equal between themselves. For, if the Tangent AD be drawn, the Rectangle under BA and AE is equal to the Square of AD; and the Rectangle under CA and AF is equal to the same Square of AD: Therefore the Rectangles shall be equal.

The END *of the* THIRD BOOK.

EUCLID's ELEMENTS.

BOOK IV.

DEFINITIONS.

I. *A Right-lined Figure is said to be inscribed in a Right-lined Figure, when every one of the Angles of the inscribed Figure touches every one of the Sides of the Figure wherein it is inscribed.*

II. *In like manner a Figure is said to be described about a Figure, when every one of the Sides of the Figure, circumscribed, touches every one of the Angles of the Figure, about which it is circumscribed.*

III. *A Right-lined Figure is said to be inscribed in a Circle, when every one of the Angles of that Figure which is inscribed, touches the Circumference of the Circle.*

IV. *A Right-lined Figure is said to be described about a Circle, when every one of the Sides of the circumscribed Figure touches the Circumference of the Circle.*

V. *So likewise a Circle is said to be inscribed in a Right-lined Figure, when the Circumference of the*

the Circle touches all the Sides of the Figure in which it is inscribed.

VI. *A Circle is said to be described about a Figure, when the Circumference of the Circle touches all the Angles of the Figure which it circumscribes.*

VII. *A Right Line is said to be applied in a Circle, when its Extremes are in the Circumference of the Circle.*

PROPOSITION I.

PROBLEM.

To apply a Right Line in a given Circle, equal to a given Right Line, whose Length does not exceed the Diameter of the Circle.

LET the Circle given be ABC, and the given Right Line, not greater than the Diameter, be D. It is required to apply a Right Line in the Circle ABC, equal to the Right Line D.

Draw BC the Diameter of the Circle; then, if BC be equal to D, what was required is done: For in the Circle ABC there is applied the Right Line BC, equal to the Right Line D: But if not, the Diameter BC is greater than D, and put * CE equal to D; and about the Centre C, with the Distance CE, let the Circle AEF be described; and join CA.

Then, because the Point C is the Centre of the Circle AEF, CA will be equal to CE; but D is equal to CE. Wherefore CA is equal to D. And so, *in the Circle* ABC, *there is applied a Right Line* CA, *equal to the given Right Line* D, *not greater than the Diameter*; which was to be done.

PROPOSITION II.

PROBLEM.

In a given Circle to inscribe a Triangle equiangular to a given Triangle.

LET ABC be a Circle given, and DEF a given Triangle. It is required to inscribe a Triangle in the Circle ABC, equiangular to the Triangle DEF.

Draw the Right Line GAH touching * the Circle * 17. 3.
ABC in the Point A; and with the Right Angle AH, at the Point A, make † an Angle HAC, equal to the † 23. 1.
Angle DEF. Likewise, at the same Point A, with the Line AG, make the Angle GAB equal to the Angle DFE; and join BC.

Then, because the Right Line HAG touches the Circle ABC, and AC is drawn from the Point of Contact in the Circle, the Angle HAC shall be ‡ equal to ‡ 32. 3.
ABC, the Angle in the alternate Segment of the Circle. But the Angle HAC is equal to the Angle DEF; therefore also the Angle ABC is equal to the Angle DEF. For the same Reason, the Angle ACB is likewise equal to the Angle DFE. Wherefore the other Angle BAC shall be † equal to the other Angle EDF. † Cor. 2.
And, consequently, *the Triangle* ABC *is equiangular to* 32. 1.
the Triangle DEF, *and is inscribed in the Circle* ABC; which was to be done.

PROPOSITION III.

PROBLEM.

About a given Circle to describe a Triangle, equiangular to a Triangle given.

LET ABC be the given Circle, and DEF the given Triangle. It is required to describe a Triangle about the Circle ABC, equiangular to the Triangle DEF.

Produce the Side EF, both Ways, to the Points G and H, and find the Centre of the Circle K, and any how draw the Line KB. Then at the Point K, with

23. 1. KB make * the Angle BKA equal to the Angle DEG; and the Angle BKC, at the same Point K on the other Side the Line KB, equal to the Angle DFH; and thro'
17. 3. the Points A, B, C, ‡ let the Right Lines LAM, MBN, NCL, be drawn, touching the Circle ABC.

Then, because the Lines LM, MN, NL, touch the Circle ABC in the Points A, B, C, and the Lines KA, KB, KC, are drawn from the Centre K to the Points
18. 3. A, B, C; the Angles at the Points A, B, C, will be † Right Angles. And because the four Angles of the quadrilateral Figure AMBK are equal to four Right Angles (for it may be divided into two Triangles), and the Angles KAM, KBM, are each Right Angles; therefore the other Angles AKB, AMB, are equal to two Right Angles. But DEG, DEF, are equal to two Right Angles; therefore the Angles AKB, AMB, are equal to the Angles DEG, DEF, whereof AKB is equal to DEG. Wherefore the other Angle AMB is equal to the other Angle DEF. In like manner we demonstrate, that the Angle LNB is equal to the Angle
‡ Cor. 2. 32. 1. DFE. Therefore the other Angle MLN is ‡ equal to the other Angle EDF. Wherefore, *the Triangle* LNM *is equiangular to the Triangle* DEF, *and is described about the Circle* ABC; which was to be done.

PROPOSITION IV.

PROBLEM.

To inscribe a Circle in a given Triangle.

LET ABC be a Triangle given. It is required to inscribe a Circle in the same.

* 9. 1. Cut * the Angles ABC, BCA, into two equal Parts by the Right Lines BD, DC, meeting each other in the Point D; and from this Point draw DE, DF,
† 12. 1. DG, † perpendicular to the Sides AB, BC, AC.

Now, because the Angle EBD is equal to the Angle FBD, and the Right Angle BED is equal to the Right Angle BFD; then the two Triangles EBD, DBF, have two Angles of the one, equal to two Angles of the other, and one Side DB common to both, *viz.* that which subtends the equal Angles; therefore the

the other Sides of the one Triangle shall be * equal to the other Sides of the other; and so DE shall be equal to DF. And, for the same Reason, DG is equal to DF; therefore DE is also equal to DG: And so the three Right Lines DE, DF, DG, are equal between themselves. Wherefore a Circle described about the Centre D, with either of the Distances DE, DF, DG, will also pass thro' the other Points. And the Sides AB, BC, AC, will touch it; because the Angles at E, F and G, are Right Angles. For if it should cut them, a Right Line, drawn on the Extremity of the Diameter of a Circle at Right Angles, will fall within the Circle; which is * absurd. Therefore a Circle described about the Centre D, with either of the Distances DE, DF, DG; will not cut the Sides AB, BC, CA; wherefore it will touch them, and will be a Circle described in the Triangle ABC. Therefore, *the Circle* EFG *is described in the given Triangle* ABC; which was to be done.

* 26. 1.

* 16. 3.

PROPOSITION V.

PROBLEM.

To describe a Circle about a given Triangle.

LET ABC be a given Triangle. It is required to describe a Circle about the same.

Bisect * the Sides AB, AC, in the Points D, E; from which Points let DF, EF, be drawn † at Right Angles to AB, AC, which will meet either within the Triangle ABC, or in the Side BC, or without the Triangle.

* 10. 1.
† 11. 1.

First, Let them meet in the Point F within the Triangle; and join BF, FC, FA. Then, because AD is equal to DB, and DF is common, and at Right Angles to AB; the Base AF will be ‡ equal to the Base FB. And after the same manner we prove, that the Base CF is equal to the Base FA. Therefore also is BF equal to CF: And so the three Right Lines FA, FB, FC, are equal to each other. Wherefore, *a Circle described about the Centre* F, *with either of the Distances*, FA, FB, FC, *will pass also thro' the other Points*,

‡

Points, and will be a Circle described about the Triangle ABC.

Secondly, Let DF, EF, meet each other in the Point F, in the Side BC, as in the second Figure; and join AF. Then we prove, as before, *that the Point* F *is the Centre of a Circle described about the Triangle* ABC.

Lastly, Let the Right Lines DF, EF, meet one another again in the Point F, without the Triangle, as in the third Figure; and join AF, FB, FC. And because AD is equal to DB, and DF is common, and at Right Angles, the Base AF shall be equal to the Base BF. So likewise we prove, that CF is also equal to AF. Wherefore BF is equal to CF. And so again, *if a Circle be described on the Centre* F, *with either of the Distances* FA, FB, FC, *it will pass thro' the other Points, and will be described about the Triangle* ABC; which was to be done.

Coroll. If a Triangle be Right-angled, the Centre of the Circle falls in the Side opposite to the Right Angle; if acute-angled, it falls within the Triangle; and if obtuse-angled, it falls without the Triangle.

PROPOSITION VI.

PROBLEM.

To inscribe a Square in a given Circle.

LET ABCD be a Circle given. It is required to inscribe a Square within the same.

Draw AC, DB, two Diameters of the Circle, cutting one another at Right Angles †; and join AB, BC, CD, DA.

Then, because BE is equal to ED (for E is the Centre), and EA is common, and at Right Angles to BD, the Base BA shall be * equal to the Base AD; and for the same Reason, BC, CD, BA, and AD, are all equal to each other. Therefore the quadrilateral Figure ABCD is equilateral. I say, it is also rectan-

gular,

gular. For, becauſe the Right Line DB is a Diameter of the Circle ABCD, therefore BAD will be a Semicircle. Wherefore the Angle BAD is * a Right Angle. * 31. 3.
And for the ſame Reaſon, every one of the other Angles ABC, BCD, CDA, is a Right Angle. Therefore ABCD is a rectangular quadrilateral Figure: But it has alſo been proved to be equilateral. Wherefore, *it ſhall neceſſarily be a Square, and is inſcribed in the Circle* ABCD; which was to be done.

PROPOSITION VII.

PROBLEM.

To deſcribe a Square about a given Circle.

LET ABCD be a Circle given. It is required to deſcribe a Square about the ſame.

Draw AC, BD, two Diameters of the Circle, cutting each other at Right Angles †; and thro' the Points † 11. 1.
A, B, C, D, draw * FG, GH, HK, KF, Tangents * 17. 3.
to the Circle ABCD.

Then, becauſe FG touches the Circle ABCD, and EA is drawn from the Centre E to the Point of Contact A, the Angles at A will be † Right Angles. For † 18. 3.
the ſame Reaſon, the Angles at the Points B, C, D, are Right Angles. And ſince the Angle AEB is a Right Angle, as alſo EBG, GH ſhall ‡ be parallel to ‡ 28. 1.
AC, and for the ſame Reaſon, AC to KF. In this manner we prove likewiſe, that GF and HK are parallel to BED; and ſo GF is parallel to HK. Therefore GK, GC, AK, FB, BK, are Parallelograms; and ſo GF is † equal to HK, and GH to FK. And ſince † 34. 1.
AC is equal to BD, and AC † equal to either GH or FK; and BD equal to either GF, or HK; GH, or FK, is equal to GF, or HK. Therefore FGHK is an equilateral quadrilateral Figure: I ſay, it is alſo equiangular. For, becauſe GBEA is a Parallelogram, and AEB is a Right Angle; then AGB ſhall be alſo a Right Angle. In like manner we demonſtrate, that the Angles at the Points H, K, F, are Right Angles. Therefore the quadrilateral Figure FGHK is rectangular; but it has been proved to be equilateral likewiſe. Wherefore, *it muſt neceſſarily be a Square, and is deſcribed about the Circle* ABCD; which was to be done.

PRO-

PROPOSITION VIII.

PROBLEM.

To describe a Circle in a given Square.

LET the given Square be ABCD. It is required to describe a Circle within the same.

* 10. 1. Bisect * the Sides AB, AD, in the Points F, E;
† 31. 1. and draw † EH thro' E, parallel to AB, or DC; and FK thro' F, parallel † to BC, or AD. Then AK, KB, AH, HD, AG, GC, BG, GD, are all Parallelo-
‡ 34. 1. grams, and their opposite Sides are ‡ equal. And because DA is equal to AB, and AE is half of AD, and AF half of AB, AE shall be equal to AF; but the opposite Sides are also equal. Therefore FG is equal to GE. In like manner we demonstrate, that GH, or GK, is equal to either FG, or GE. Therefore GE, GF, GH, GK, are equal to each other: And so a Circle, being described about the Centre G, with either of the Distances GE, GF, GH, GK, will also pass thro' the other Points, and shall touch the Sides DA, AB, BC, CD; because the Angles at E, F, H, K, are Right Angles. For if the Circle should cut the Sides of the Square, a Right Line, drawn from the End of the Diameter of a Circle, at Right Angles, will fall within the
* 16. 3. Circle; which is * absurd. Wherefore a Circle described about the Centre G, with either of the Distances GE, GF, GH, GK, will not cut DA, AB, BC, CD, the Sides of the Square. Wherefore, *it shall necessarily touch them, and will be described in the Square* ABCD; which was to be done.

PROPOSITION IX.

PROBLEM.

To describe a Circle about a Square given.

LET ABCD be a Square given. It is required to circumscribe a Circle about the same.

Join AC, BD, mutually cutting one another in the Point E.

And

And since DA is equal to AB, and AC is common, the two Sides DA, AC, are equal to the two Sides BA, AC; but the Base DC is equal to the Base BC. Therefore the Angle DAC will * be equal to the Angle BAC: * 8. 1.
And consequently the Angle DAB is bisected by the Right Line AC. In the same manner we prove, that each of the other Angles ABC, BCD, CDA, are bisected by the Right Lines AC, DB.

Then, because the Angle DAB is equal to the Angle ABC, and the Angle EAB is half of the Angle DAB, and the Angle EBA half of the Angle ABC; the Angle EAB shall be equal to the Angle EBA: And so the Side EA is † equal to the Side EB. In like † 6. 1.
manner we demonstrate, that each of the Right Lines EC, ED, is equal to each of the Right Lines EA, EB. Therefore the four Right Lines EA, EB, EC, ED, are equal between themselves. Wherefore, *a Circle being described about the Centre* E, *with either of the Distances* EA, EB, EC, ED, *will also pass thro' the other Points, and will be described about the Square* ABCD; which was to be done.

PROPOSITION X.

PROBLEM.

To make an Isosceles Triangle, having each of the Angles at the Base double to the other Angle.

CUT * any given Right Line AB in the Point C, * 11. 2.
so that the Rectangle contained under AB and BC be equal to the Square of AC; then about the Centre A, with the Distance AB, let the Circle BDE be described; and † in the Circle BDE apply the Right Line † 1 of this.
BD equal to AC; which is not greater than the Diameter. This being done, join DA, DC, and describe ‡ a Circle ACD about the Triangle ADC. ‡ 5 of this.

Then, because the Rectangle under AB and BC is equal to the Square of AC, and AC is equal to BD, the Rectangle under AB and BC shall be equal to the Square of BD. And because some Point B, is taken without the Circle ACD, and from that Point there fall two Right Lines, BCA, BD, to the Circle, one of which

cuts

cuts the Circle, and the other falls on it; and since the Rectangle under AB and BC is equal to the Square of
* 37. 3. BD, the Right Line BD shall * touch the Circle ACD. And since BD touches it, and DC is drawn from the Point of Contact D, the Angle BDC is equal to the Angle in the alternate Segment of the Circle, *viz.* equal
† 32. 3. † to the Angle DAC. And since the Angle BDC is equal to the Angle DAC; if CDA, which is common, be added, the whole Angle BDA is equal to the two Angles CAD, DAC. But the outward Angle BCD
‡ 32. 1. is ‡ equal to CDA and DAC. Therefore BDA is
* 5. 1. equal to BCD. But the Angle BDA * is equal to the Angle CBD, because the Side AD is equal to the Side AB. Wherefore DBA shall be equal to BCD: And so the three Angles BDA, DBA, BCD, are equal to each other. And since the Angle DBC is equal to the
† 6. 1. Angle BCD, the Side BD is † equal to the Side DC. But BD is put equal to CA. Therefore CA is equal
‡ 5. 1. to CD. And so the Angle CDA is ‡ equal to the Angle DAC. Therefore the Angles CDA, DAC, taken together, are double to the Angle DAC. But the Angle BCD is equal to the Angles CDA, and DAC. Therefore the Angle BCD is double to the Angle DAC. But BCD is equal to BDA, or DBA. Wherefore BDA, or DBA, is double to DAB. Therefore, *the Isosceles Triangle* ABD *is made, having each of the Angles at the Base double to the other Angle;* which was to be done.

PROPOSITION XI.

PROBLEM.

To describe an equilateral and equiangular Pentagon in a given Circle.

LET ABCDE be a Circle given. It is required to describe an equilateral and equiangular Pentagon in the same.

* 10 *of this.* Make an Isosceles Triangle FGH, having * each of the Angles at the Base GH, double to the other Angle F; and describe the Triangle ADC in the Circle
† 2 *of this.* ABCDE, equiangular † to the Triangle FGH; so that

that the Angle CAD be equal to that at F, and ACD, CDA, each equal to the Angles G or H. Wherefore the Angles ACD, CDA, are each double to the Angle CAD. This being done, bisect the Angles * ACD, CDA, by the Right Lines CE, DB, and join AB, BC, DE, EA. * 9. 1.

Then, because each of the Angles ACD, CDA, is double to CAD, and they are bisected by the Right Lines CE, DB; the five Angles DAC, ACE, ECD, CDB, BDA are equal to each other. But equal Angles stand † upon equal Circumferences. Therefore the five Circumferences AB, BC, CD, DE, EA, are equal to each other. But equal Circumferences subtend ‡ equal Right Lines. Therefore the five Right Lines AB, BC, CD, DE, EA, are equal to each other. Wherefore ABCDE is an equilateral Pentagon. I say, it is also equiangular: For because the Circumference AB is equal to the Circumference DE; by adding the Circumference BCD, which is common, the whole Circumference ABCD is equal to the whole Circumference EDCB: But the Angle AED stands on the Circumference ABCD, and BAE on the Circumference EDCB; therefore the Angle BAE is equal to the Angle AED. For the same Reason, each of the other Angles ABC, BCD, CDE, is equal to BAE, or AED; wherefore the Pentagon ABCDE is equiangular. But it has been proved to be also equilateral: And, consequently, *there is an equilateral and equiangular Pentagon inscribed in a given Circle*; which was to be done.

† 26. 3.

‡ 29. 3.

PROPOSITION XII.

PROBLEM.

To describe an equilateral and equiangular Pentagon about a Circle given.

LET ABCDE be the given Circle. It is required to describe an equilateral and equiangular Pentagon about the same.

Let A, B, C, D, E, be the angular Points of a Pentagon supposed to be inscribed * in the Circle; so that the Circumferences AB, BC, CD, DE, EA, be equal;

* By 11 of this.

equal; and let the Right Lines GH, HK, KL, LM, MG, be drawn, touching † the Circle in the Points A, B, C, D, E: Let F be the Centre of the Circle ABCDE; and join FB, FK, FC, FL, FD.

† 17. 3.

Then, because the Right Line KL touches the Circle ABCDE in the Point C, and the Right Line FC is drawn from the Centre F to C, the Point of Contact; FC will be ‡ perpendicular to KL; and so both the Angles at C are Right Angles. For the same Reason, the Angles at the Points B, D, are Right Angles. And because FCK is a Right Angle, the Square of FK will be * equal to the Squares of FC, CK: And for the same Reason, the Square of FK is equal to the Squares of FB, BK. Therefore the Squares of FC, CK, are equal to the Squares of FB, BK. But the Square of FC is equal to the Square of FB: Wherefore the Square of CK shall be equal to the Square of BK; and so BK is equal to CK. And because FB is equal to FC, and FK is common, the two Sides BF, FK, are equal to the two Sides CF, FK, and the Base BK is equal to the Base KC; and so the Angle BFK shall be † equal to the Angle KFC, and the Angle BKF to the Angle FKC. Therefore the Angle BFC is double to the Angle KFC, and the Angle BKC double to the Angle FKC: For the same Reason the Angle CFD is double to the Angle CFL, and the Angle CLD double to the Angle CLF. And because the Circumference BC is equal to the Circumference CD, the Angle BFC shall be ‡ equal to the Angle CFD. But the Angle BFC is double to the Angle KFC, and the Angle DFC double to LFC. Therefore the Angle KFC is equal to the Angle CFL. And so FKC, FLC, are two Triangles, having two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, *viz.* the common Side FC; wherefore they shall have † the other Sides of the one equal to the other Sides of the other; and the other Angle of the one equal to the other Angle of the other. Therefore the Right Line KC is equal to the Right Angle CL, and the Angle FKC to the Angle FLC. And since KC is equal to CL, KL shall be double to KC. And by the same Reason, we prove that HK is double to BK. Again, because BK has been

‡ 18. 3.

47. 1.

† 8. 1.

‡ 27. 3.

† 26. 1.

been proved equal to KC, and KL the double of KC, as also HK the double of BK; HK shall be equal to KL. So likewise we prove, that GH, GM and ML, are each equal to HK, or KL: Therefore the Pentagon GHKLM is equilateral. I say, also it is equiangular. For because the Angle FKC is equal to the Angle FLC, and the Angle HKL has been proved to be double to the Angle FKC; and also KLM double to FLC: Therefore the Angle HKL shall be equal to the Angle KLM. By the same Reason we demonstrate, that every one of the Angles KHG, HGM, GML, is equal to the Angle HKL, or KLM. Therefore the five Angles, GHK, HKL, KLM, LMG, MGH, are equal between themselves. And so, *the Pentagon* GHKLM *is equiangular; and it has been proved likewise to be equilateral, and described about the Circle* ABCDE; which was to be done.

PROPOSITION XIII.

Problem.

To describe a Circle in an equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to inscribe a Circle in the same.

Bisect * the Angles BCD, CDE, by the Right Lines CF, DF; and from the Point F, wherein CF, DF, meet each other, let the Right Lines FB, FA, FE, be drawn. Now, because BC is equal to CD, and CF is common, the two Sides BC, CF, are equal to the two Sides DC, CF; and the Angle BCF is equal to the Angle DCF. Therefore the Base BF is † equal to the Base FD; and the Triangle BFC equal to the Triangle DCF, and the other Angles of the one equal to the other Angles of the other, which are subtended by the equal Sides: Therefore the Angle CBF shall be equal to the Angle CDF. And because the Angle CDE is double to the Angle CDF, and the Angle CDE is equal to the Angle ABC, as also CDF equal to CBF; the Angle CBA will be double to the Angle CBF; and so the Angle ABF equal to the Angle CBF:

* 9. 1.

† 4. 1.

CBF: Wherefore the Angle ABC is bisected by the Right Line BF. After the same manner we prove, that either of the Angles BAE, or AED, is bisected by the Right Line AF, or FE. From the Point F draw
* 12. 1. * FG, FH, FK, FL, FM, perpendicular to the Right Lines AB, BC, CD, DE, EA: Then, since the Angle HCF is equal to the Angle KCF, and the Right Angle FHC equal to the Right Angle FKC; the two Triangles FHC, FKC, shall have two Angles of the one equal to two Angles of the other, and one Side of the one equal to one Side of the other, *viz.* the Side FC common to each of them: And so the other Sides
† 26. 1. of the one will be † equal to the other Sides of the other, and the Perpendicular FH equal to the Perpendicular FK. In the same manner we demonstrate, that FL, FM, or FG, is equal to FH, or FK: Therefore the five Right Lines FG, FH, FK, FL, FM, are equal to each other, and so a Circle described on the Centre F, with either of the Distances FG, FH, FK, FL, FM, will pass thro' the other Points, and shall touch the Right Lines AB, BC, CD, DE, EA; since the Angles at G, H, K, L, M, are Right Angles. For, if it does not touch them but cuts them, a Right Line drawn from the Extremity of the Diameter of a Circle, at Right Angles to the Diameter, will fall within
‡ 16. 3. the Circle; which is ‡ absurd. Therefore, *a Circle described on the Centre* F, *with the Distance of any one of the Points* G, H, K, L, M, *will not cut the Right Lines* AB, BC, CD, DE, EA, *and so will necessarily touch them*; which was to be done.

Coroll. If two of the nearest Angles of an equilateral and equiangular Figure be bisected, and, from the Point in which the Lines bisecting the Angles meet, there be drawn Right Lines to the other Angles of the Figure, all the Angles of the Figure will be bisected.

PRO-

PROPOSITION XIV.

PROLBEM.

To describe a Circle about a given equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to describe a Circle about the same.

Bisect both the Angles BCD, CDE, by the Right Lines CF, FD; and draw FB, FA, FE, from the Point F, in which they meet. Then each of the other Angles CBA, BAE, AED, shall be bisected* by the Right Lines BF, FA, FE. And since the Angle BCD is equal to the Angle CDE, and the Angle FCD is half the Angle BCD; as likewise CDF, half CDE; the Angle FCD will be equal to the Angle FDC; and so the Side CF † equal to the Side FD. We demonstrate, in like manner, that FB, FA, or FE, is equal to FC, or FD. Therefore the five Right Lines FA, FB, FC, FD, FE, are equal to each other. And so *a Circle being described on the Centre* F, *with any of the Distances* FA, FB, FC, FD, FE, *will pass thro' the other Points, and will be described about the equilateral and equiangular Pentagon* ABCDE; which was to be done.

* Cor. of preced.

† 6. 1.

PROPOSITION XV.

PROLBEM.

To inscribe an equilateral and equiangular Hexagon in a given Circle.

LET ABCDEF be a Circle given. It is required to inscribe an equilateral and equiangular Hexagon therein.

Draw AD, a Diameter of the Circle ABCDEF, and let G be the Centre; and about the Point D, as a Centre, with the Distance DG, let a Circle, EGCH, be described; join EG, GC, which produce to the Points B, F: Likewise join AB, BC, CD, DE, EF,

FA: I say, ABCDEF is an equilateral and equiangular Hexagon.

For, since the Point G is the Centre of the Circle ABCDEF, GE will be equal to GD. Again, because the Point D is the Centre of the Circle EGCH, DE shall be equal to DG: But GE has been proved equal to GD; therefore GE is equal to ED. And so EGD is an equilateral Triangle; and consequently the three
* Cor. 5. 1. Angles thereof, EGD, GDE, DEG, are * equal between themselves. But the three Angles of a Triangle
† 32. 1. are † equal to two Right Angles; therefore the Angle EGD is a third Part of two Right Angles. In the same manner we demonstrate, that DGC is one third Part of two Right Angles: And since the Right Line
‡ 13. 1. CG, standing upon the Right Line EB, makes ‡ the adjacent Angles EGC, CGB; therefore the other Angle, CGB, is also one third Part of two Right Angles. Therefore the Angles EGD, DGC, CGB, are equal between themselves: And the Angles that are vertical to them, *viz.* the Angles BGA, AGF,
* 15. 1. FGE, are * equal to the Angles EGD, DGC, CGB. Wherefore the six Angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. But equal An-
† 26. 3. gles stand † on equal Circumferences: Therefore the six Circumferences AB, BC, CD, DE, EF, FA, are
‡ 29. 3. equal to each other. But equal Right Lines subtend ‡ equal Circumferences: Therefore the six Right Lines are equal between themselves; and accordingly the Hexagon ABCDEF is equilateral. I say, it is also equiangular: For, because the Circumference AF is equal to the Circumference ED, add the common Circumference ABCD, and the whole Circumference FABCD is equal to the whole Circumference EDCBA. But the Angle FED stands on the Circumference FABCD; and the Angle AFE, on the Circumference
* 27. 3. EDCBA: Therefore the Angle AFE is * equal to the Angle DEF. In the same manner we prove, that the other Angles of the Hexagon ABCDEF are severally equal to AFE, or FED. Therefore, *the Hexagon* ABCDEF *is equiangular. But it has been proved to be also equilateral, and is inscribed in the Circle* ABCDEF; which was to be done.

Coroll.

Coroll. From hence it is manifeſt, that the Side of the Hexagon is equal to the Semidiameter of the Circle. And if we draw, thro' the Points A, B, C, D, E, F, Tangents to the Circle, an equilateral and equiangular Hexagon, will be deſcribed about the Circle, as is manifeſt, from what has been ſaid concerning the Pentagon. And ſo likewiſe may a Circle be inſcribed and circumſcribed about a given Hexagon; *which was to be done.*

PROPOSITION XVI.

PROBLEM.

To deſcribe an equilateral and equiangular Quindecagon in a given Circle.

LET ABCD be a Circle given. It is required to deſcribe an equilateral and equiangular Quindecagon in the ſame.

Let AC be the Side of an equilateral Triangle inſcribed in the Circle ABCD, and AB the Side of a Pentagon. Now, if the whole Circumference of the Circle ABCD be divided into fifteen equal Parts, the Circumference ABC, one Third of the Whole, ſhall be five of the ſaid fifteen equal Parts; and the Circumference AB, one Fifth of the Whole, will be three of the ſaid Parts: Wherefore the remaining Circumference BC will be two of the ſaid Parts. And if BC be biſected in the Point E, then BE, or EC, will be one fifteenth Part of the whole Circumference ABCD. And ſo, *if* BE, EC, *be joined, and either* EC, *or* EB, *be continually applied in the Circle, there ſhall be an equilateral and equiangular Quindecagon deſcribed in the Circle* ABCD; which was to be done.

If, according to what hath been ſaid of the Pentagon, Right Lines are drawn thro' the Diviſions of the Circle touching the ſame, there will be deſcribed about the Circle an equilateral and equiangular Quindecagon. And, moreover, a Circle may be inſcribed, or circumſcribed, about a given equilateral and equiangular Quindecagon.

EUCLID's ELEMENTS.

BOOK V.

DEFINITIONS.

I. *A Part is a Magnitude of a Magnitude, the Less of the Greater, when the Lesser measures the Greater.*

II. *But a Multiple is a Magnitude of a Magnitude, the Greater of the Lesser, when the Lesser measures the Greater.*

III. *Ratio is a certain mutual Habitude of Magnitudes of the same Kind, according to Quantity.*

IV. *Magnitudes are said to have Proportion to each other, which, being multiplied, can exceed one another.*

V. *Magnitudes are said to be in the same Ratio, the first to the second, and the third to the fourth; when the Equimultiples of the first and third, compared with the Equimultiples of the second and fourth, according to any Multiplication whatsoever, are either both together, greater, equal, or less, than the Equimultiples of the second and fourth, if those be taken that answer each other.*

That

That is, if there be four Magnitudes, and you take any Equimultiples of the firſt and third, and alſo any Equimultiples of the ſecond and fourth; and if the Multiple of the firſt be greater than the Multiple of the ſecond; and alſo the Multiple of the third greater than the Multiple of the fourth; or, if the Multiple of the firſt be equal to the Multiple of the ſecond; and alſo the Multiple of the third equal to the Multiple of the fourth; or, laſtly, if the Multiple of the firſt be leſs than the Multiple of the ſecond; and alſo that of the third leſs than that of the fourth, and theſe Things happen according to every Multiplication whatſoever: Then the four Magnitudes are in the ſame Ratio; the firſt to the ſecond, as the third to the fourth.

VI. *Magnitudes that have the ſame Proportion, are called Proportionals.*

Expounders uſually lay down here that Definition, for Magnitudes, which *Euclid* has given for Numbers, only, in this Seventh Book; *viz.* That

Numbers are proportional, when the firſt is either the ſame Multiple of the ſecond, as the third is of the fourth, or elſe the ſame Part or Parts.

But this Definition appertains only to Numbers, and commenſurable Quantities; and ſo, ſince it is not univerſal, *Euclid* did well to reject it in this Element, which treats of the Properties of all Proportionals; and to ſubſtitute another general one, agreeing to all Kinds of Magnitudes. In the mean Time, Expounders very much endeavour to demonſtrate the Definition here laid down by *Euclid*, by the uſual received Definition of proportional Numbers; but this much eaſier flows from that, than that from this; which may be thus demonſtrated:

Firſt, Let A, B, C, D, be four Magnitudes, which are in the ſame Ratio, according to the Conditions that Magnitudes in the ſame Ratio muſt have according to the fifth Definition; and let the firſt be a Multiple of the ſecond: I ſay, the third is alſo the ſame Multiple of the fourth. For Example: Let A be equal to 5B: Then C ſhall be equal to 5D. Take

any Number, for Example, 2, by which let 5 be multiplied, and the Product will be 10: And let 2A, 2C, be Equimultiples of the first and third Magnitudes A and C: Also, let 10B and 10D be Equimultiples of the second and fourth Magnitudes B and D. Then (by *Def.* 5.) if 2A be equal to 10B, 2C shall be equal to 10D. But since A (from the *Hypothesis*) is five Times B, 2A shall be equal to 10B; and so 2C equal to 10D, and C equal to 5D; that is, C will be five Times D. W. W. D.

A : B : : C : D

2A, 10B, 2C, 10D

Secondly, Let A be any Part of B; then C will be the same Part of D. For, because A is to B, as C is to D; and since A is some Part of B; then B will be a Multiple of A: And so (by *Case* 1.) D will be the same Multiple of C; and accordingly C shall be the same Part of the Magnitude D, as A is of B. W. W. D.

Thirdly, Let A be equal to any Number of whatsoever Parts of B. I say, C is equal to the same Number of the like Parts of D. For Example: Let A be a fourth Part of five Times B; that is, let A be equal to $\frac{5}{4}$B. I say, C is also equal to $\frac{5}{4}$D. For, because A is equal $\frac{5}{4}$B, each of them being multiplied by 4, then 4A will be equal to 5B. And so, if the Equimultiples of the first and third, *viz.* 4A, 4C, be assumed; as also the Equimultiples of the second and fourth, *viz.* 5B, 5D; and (by the *Definition*) if 4A is equal to 5B; then 4C is equal to 5D. But 4A has been proved equal to 5B, and so 4C shall be equal to 5D, and C equal to $\frac{5}{4}$D. W. W. D.

A : B : : C : D

4A, 5B, 4C, 5D

And universally, if A be equal to $\frac{n}{m}$B, C will be equal to $\frac{n}{m}$D. For let A and C be multiplied by m, and B and D by n. And because A is equal to $\frac{n}{m}$B; mA shall be equal to nB; wherefore (by *Def.* 5.) mC will be equal to nD, and C equal to $\frac{n}{m}$D. W. W. D.

A : B : : C : D

mA, nB, mC, nD

VII. *When*

VII. *When, of Equimultiples, the Multiple of the first exceeds the Multiple of the second, but the Multiple of the third does not exceed the Multiple of the fourth; then the first to the second is said to have a greater Proportion, than the third to the fourth.*

VIII. *Analogy is a Similitude of Proportions.*

IX. *Analogy at least consists of three Terms.*

X. *When three Magnitudes are Proportionals, the first is said to have, to the third, a duplicate Ratio to what it has to the second.*

XI. *But when four Magnitudes are continued Proportionals, the first shall have a triplicate Ratio to the fourth of what it has to the second; and so always one more in Order, as the Proportionals shall be extended.*

XII. *Homologous Magnitudes, or Magnitudes of a like Ratio, are said to be such whose Antecedents are to the Antecedents, and Consequents to the Consequents.*

XIII. *Alternate Ratio is the comparing of the Antecedent with the Antecedent, and the Consequent with the Consequent.*

XIV. *Inverse Ratio is, when the Consequent is taken as the Antecedent, and so compared with the Antecedent as a Consequent.*

XV. *Compounded Ratio is, when the Antecedent and Consequent, taken both as one, is compared to the Consequent itself.*

XVI. *Divided Ratio is, when the Excess, whereby the Antecedent exceeds the Consequent, is compared with the Consequent.*

XVII. *Converse Ratio is, when the Antecedent is compared with the Excess, by which the Antecedent exceeds the Consequent.*

XVIII. *Ratio of Equality is, where there are taken more than two Magnitudes in one Order, and a like*

like Number of Magnitudes in another Order, comparing two to two being in the same Proportion; and it shall be in the first Order of Magnitudes, as the first is to the last, so in the second Order of Magnitudes is the first to the last: Or otherwise, it is the Comparison of the Extremes together, the Means being omitted.

XIX. *Ordinate Proportion is, when as the Antecedent is to the Consequent, so is the Antecedent to the Consequent; and as the Consequent is to any other, so is the Consequent to any other.*

XX. *Perturbate Proportion is, when there are three or more Magnitudes, and others also, that are equal to these in Multitude, as in the first Magnitudes the Antecedent is to the Consequent; so in the second Magnitudes is the Antecedent to the Consequent: And as in the first Magnitudes the Consequent is to some other; so in the second Magnitudes is some other, to the Antecedent.*

AXIOMS.

I. *EQuimultiples of the same, or of equal Magnitudes, are equal to each other.*

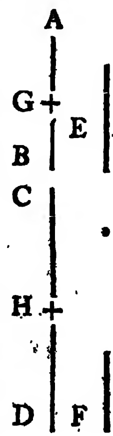
II. *Those Magnitudes that have the same Equimultiple, or whose Equimultiples are equal, are equal to each other.*

PROPOSITION I.

Theorem.

If there be any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each of each; whatſoever Multiple any one of the former Magnitudes is of its correſpondent one, the ſame Multiple are all the former Magnitudes of all the latter.

LET there be any Number of Magnitudes AB, CD Equimultiples of a like Number of Magnitudes E, F, each of each. I ſay, what Multiple the Magnitude AB is of E, the ſame Multiple AB, and CD, together, is of E and F together.



For, becauſe AB and CD are Equimultiples of E and F, as many Magnitudes equal to E, that are in AB, ſo many ſhall be equal to F in CD. Now, divide AB into Parts equal to E, which let be AG, GB; and CD into Parts equal to F, *viz.* CH, HD. Then the Multitude of Parts, CH, HD, ſhall be equal to the Multitude of Parts AG, GB. And ſince AG is equal to E, and CH to F; AG and CH, together, ſhall be equal to E and F together. By the ſame Reaſon, becauſe GB is equal to E, and HD to F, GB and HD, together, will be equal to E and F together. Therefore, as often as E is contained in AB, ſo often is E and F, together, contained in AB and CD, together. And ſo as often as F is contained in CD, ſo often are E and F, together, contained in AB, and CD together. Therefore, *if there are any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each of each; whatſoever Multiple any one of the former Magnitudes is of its correſpondent one, the ſame Multiple are all the former Magnitudes of all the latter*; which was to be demonſtrated.

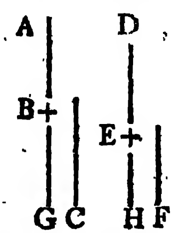
PRO-

PROPOSITION II.

Theorem.

If the first be the same Multiple of the second, as the third is of the fourth; and if the fifth be the same Multiple of the second, as the sixth is of the fourth; then shall the first, added to the fifth, be the same Multiple of the second, as the third, added to the sixth, is of the fourth.

LET the first AB be the same Multiple of the second C, as the third DE is of the fourth F; and let the fifth BG be the same Multiple of the second C, as the sixth EH is of the fourth F. I say, the first added to the fifth, *viz.* AG, is the same Multiple of the second C, as the third added to the sixth, *viz.* DH, is of the fourth F.



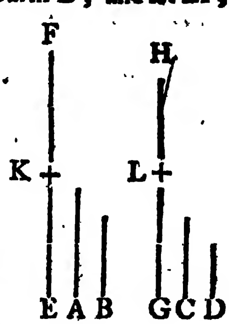
For, because AB is the same Multiple of C, as DE is of F; there are as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. And, for the same Reason, there are as many Magnitudes equal to C in BG, as there are Magnitudes equal to F in EH. Therefore there are as many Magnitudes equal to C, in the whole AG, as there are Magnitudes equal to F in DH. Wherefore AG is the same Multiple of C, as DH is of F. And so the first, added to the fifth, AG, is the same Multiple of the second C, as the third, added to the sixth, DH, is of the fourth F. Therefore, *if the first be the same Multiple of the second, as the third is of the fourth; and if the fifth be the same Multiple of the second, as the sixth is of the fourth; then shall the first, added to the fifth, be the same Multiple of the second, as the third, added to the sixth, is of the fourth*; which was to be demonstrated.

PROPOSITION III.

Theorem.

If the first be the same Multiple of the second, as the third is of the fourth, and there be taken Equimultiples of the first and third; then will the Magnitudes so taken be Equimultiples of the second and fourth.

LET the first A be the same Multiple of the second B, as the third C is of the fourth D; and let EF, GH, be Equimultiples of A and C. I say, EF, is the same Multiple of B as GH is of D.



For, because EF is the same Multiple of A, as GH is of C, there are as many Magnitudes equal to A in EF, as there are Magnitudes equal to C in GH. Now divide EF into the Magnitudes EK, KF, each equal to A, and GH into the Magnitudes GL, LH, each equal to C. Then the Number of the Magnitudes EK, KF, will be equal to the Number of the Magnitudes GL, LH. And because A is the same Multiple of B, as C is of D, and EK is equal to A, and GL to C; EK will be the same Multiple of B, as GL is of D. For the same Reason, KF shall be the same Multiple of B, as LH is of D. Therefore because the first EK is the same Multiple of the second B, as the third GL is of the fourth D, and KF the fifth, is the same Multiple of B, the second, that LH, the sixth, is of D the fourth: Therefore the first added to the fifth, EF, shall be * the same Multiple of the second B, as the third added to the sixth, GH, is of the fourth D. *If* therefore, *the first be the same Multiple of the second, as the third is of the fourth, and there be taken Equimultiples of the first and third; then will the Magnitudes so taken, be Equimultiples of the second and fourth*; which was to be demonstrated.

* 2 of this.

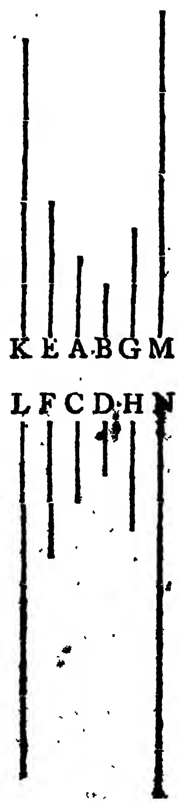
PRO-

PROPOSITION IV.

Theorem.

If the first have the same Proportion to the second, as the third to the fourth; then also shall the Equimultiples of the first and third have the same Proportion to the Equimultiples of the second and fourth, according to any Multiplication whatsoever, if they be so taken as to answer each other.

LET the first A have the same Proportion to the second B, as the third C hath to the fourth D; and let E and F, the Equimultiples of A and C, be any how taken; as also G and H, the Equimultiples of B and D. I say, E is to G as F is to H.



For take K and L, any Equimultiples of E and F; and also M and N, any, of G and H.

Then, because E is the same Multiple of A, as F is of C, and K and L are taken Equimultiples of E
* 3 *of this.* and F; therefore K will be * the same Multiple of A, as L is of C. For the same Reason, M is the same Multiple of B, as N is of D. And since A is to B, as C is to D, and K and L are taken Equimultiples of A and C; and also M and N Equimultiples of B and D; if K exceeds
† *Def. of this.* M, then † L will exceed N; if K be equal to M, L will be equal to N; and if K is less than M, then L will be less than N: But K and L are Equimultiples of E and F, also M and N are Equimultiples of G and H. Therefore, as
‡ *Def.* 5. E is to G, so shall ‡ F be to H. Wherefore, *if the first have the same Proportion to the second, as the third to the fourth; then also shall the Equimultiples*

multiples of the first and third have the same Proportion to the Equimultiples of the second and fourth, according to any Multiplication whatsoever, if they be so taken as to answer each other; which was to be demonstrated.

Because it is demonstrated, if K exceeds M, then L will exceed N; and if K be equal to M, L will be equal to N; and if, K be less than M, L will be less than N: It is manifest, likewise, if M exceeds K, that N shall exceed L; if equal, equal; but if less less. And therefore, as G is to E, so is * H to F. * *Def.* 5.

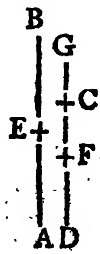
Coroll. From hence it is manifest, if four Magnitudes be proportional, that they will be also inversely proportional.

PROPOSITION V.

THEOREM.

If one Magnitude be the same Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole.

LET the Magnitude AB be the same Multiple of the Magnitude CD, as the Part taken away AE, is of the Part taken away CF. I say, that the Residue EB is the same Multiple of the Residue FD, as the Whole AB is of the Whole CD.



For, let EB be such a Multiple of CG, as AE is of CF.

Then, because AE is the same Multiple of CF, as EB is of CG, AE will be * the same Multiple of CF, as AB is of GF. But AE and AB are put Equimultiples of CF and CD: Therefore AB is the same Multiple of GF, as of CD; and so GF is † equal to CD. Now, let CF, which is common, be taken away; then the Residue GC is equal to the Residue

* 1 *of this.*
† *Ax.* 2. *of this.*

sidue DF. And then, because AE is the same Multiple of CF, as EB is of CG, and CG is equal to DF; AE shall be the same Multiple of CF, as EB is of FD. But AE is put the same Multiple of CF, as AB is of CD: Therefore EB is the same Multiple of FD, as AB is of CD; and so the Residue EB is the same Multiple of the Residue FD, as the Whole AB is of the Whole CD. Wherefore, *if one Magnitude be the same Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole*; which was to be demonstrated.

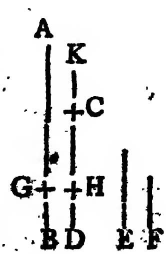
PROPOSITION VI.

Theorem.

If two Magnitudes be Equimultiples of two Magnitudes, and some Magnitudes Equimultiples of the same, be taken away; then the Residues are either equal to those Magnitudes, or else Equimultiples of them.

LET two Magnitudes AB, CD, be Equimultiples of two Magnitudes E, F; and let the Magnitudes AG, CH, Equimultiples of the same, E, F, be taken from AB, CD: I say, the Residues GB, HD, are either equal to E, F, or are Equimultiples of them.

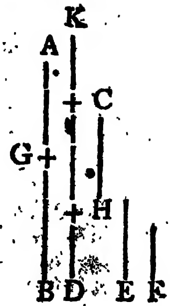
For, first, Let GB be equal to E. I say, HD is also equal to F. For let CK be equal to F. Then, because AG is the same Multiple of E, as CH is of F; and GB is equal to E; and CK to F; AB will be * the same Multiple of E, as KH is of F. But AB and CD are put Equimultiples of E and F. Therefore KH is the same Multiple of F, as CD is of F.



* 1 of this.

And because KH and CD are Equimultiples of F; KH will be equal to CD. Take away

A



away CH, which is common; then the Refidue KC is equal to the Re-fidue HD. But KC is equal to F. Therefore HD is equal to F; and fo GB fhall be equal to E, and HD to F.

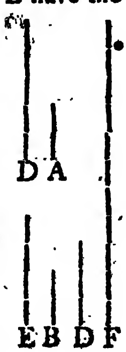
In like manner we demonftrate, if GB was a Multiple of E, that HD is a like Multiple of F. Therefore, *if two Magnitudes be Equimultiples of two Magnitudes, and fome Magnitudes, Equimultiples of the fame, be taken away; then the Refidues are either equal to thofe Magnitudes, or elfe Equimultiples of them;* which was to be demonftrated.

PROPOSITION VII.

Theorem.

Equal Magnitudes have the fame Proportion to the fame Magnitude; and one and the fame Magnitude has the fame Proportion to equal Magnitudes.

LET A, B, be equal Magnitudes, and let C be any other Magnitude. I fay, A and B have the fame Proportion to C; and likewife C has the fame Proportion to A as to B.



For take D, and E, Equimultiples of A and B; and let F be any other Multiple of C.

Now, becaufe D is the fame Multiple of A, as E is of B, and A is equal to B, D fhall be alfo equal to E; but F is a Magnitude taken at Pleafure. Therefore if D exceeds F, then E will exceed F; if D be equal to F, E will be equal to F; and if lefs, lefs. But D and E are Equimultiples of A and B; and F is any other Multiple of C. Therefore it will be * as A is to C, fo is B to C. * Def. 5.

I say, moreover, that C has the same Proportion to A as to B. For the same Construction remaining, we prove, in like manner, that D is equal to E. Therefore, if F exceeds D, it will also exceed E; if it be equal to D, it will be equal to E; and if it be less than D, it will be less than E. But F is a Multiple of C; and D and E, any other Equimultiples of A and B;
Def. 5. therefore, as C is to A, so shall * C be to B. Wherefore, *equal Magnitudes have the same Proportion to the same Magnitude, and the same Magnitude to equal ones*; which was to be demonstrated.

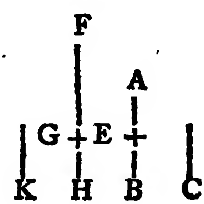
PROPOSITION VIII.

THEOREM.

The greater of any two unequal Magnitudes has a greater Proportion to some third Magnitude, than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes, than it has to the greater.

LET AB and C be two unequal Magnitudes, whereof AB is the greater; and let D be any third Magnitude. I say, AB has a greater Proportion to D, than C has to D; and D has a greater Proportion to C, than it has to AB.

Because AB is greater than C, make BE equal to C, that is, let AB exceed C by AE; then AE, multiplied some Number of Times, will be greater than D. Now let AE be multiplied until it exceeds D, and let that Multiple of AE greater than D be FG. Make GH the same Multiple of EB, and K of C, as FG is of AE. Also, assume L double to D, P triple, and so on, until such a Multiple of D is had, as is the nearest greater than K; let this be N, and let M be a Multiple of D, the nearest less than N.



Now, because N is the nearest Multiple of D greater

than

than K; M will not be greater than K; that is, K will not be less than M. And since FG is the same Multiple of AE, as GH is of EB; FG shall be * the same Multiple of AE, as FH is of AB; but FG is the same Multiple of AE, as K is of C: Wherefore FH is the same Multiple of AB as K is of C; that is, FH and K are Equimultiples of AB and C. Again, because GH is the same Multiple of EB, as K is of C, and EB is equal to C; GH shall be † equal to K. But K is not less than M: Therefore GH shall not be less than M. But FG is greater than D: Therefore the whole FH will be greater than M and D; but M and D, together are equal to N, because M is a Multiple of D, the nearest lesser than N: Wherefore FH is greater than N. And so, since FH exceeds N, and K does not; and FH and K are Equimultiples of AB and C, and N is another Multiple of D; therefore AB will have ‡ a greater Ratio to D, than C has to D. I say, moreover, that D has a greater Ratio to C than it has to AB: For the same Construction remaining, we demonstrate, as before, that N exceeds K, but not FH. And N is a Multiple of D, and FH and K are Equimultiples of AB and C. Therefore D has ‡ a greater Proportion to C, than D hath to AB. Wherefore, *the greater of any two unequal Magnitudes has a greater Proportion to some third Magnitude than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes than it has to the greater*; which was to be demonstrated.

* 1 of this.

† Ax. 1.

‡ Def. 7.

PROPOSITION IX.

THEOREM.

Magnitudes which have the same Proportion to one and the same Magnitude, are equal to one another; and if a Magnitude has the same Proportion to other Magnitudes, these Magnitudes are equal to one another.

LET the Magnitudes A and B have the same Proportion to C. I say, A is equal to B.

For,

* 8 of this. For, if it was not, A and B would not * have the same Proportion to the same Magnitude C; but they have. Therefore A is equal to B.

Again, let C have the same Proportion to A as to B. I say, A is equal to B.

For, if it be not, C will not * have the same Proportion to A as to B; but it hath: Therefore A is necessarily equal to B. Therefore, *Magnitudes that have the same Proportion to one and the same Magnitude, are equal to one another; and, if a Magnitude, has the same Proportion to other Magnitudes, these Magnitudes are equal to one another*; which was to be demonstrated.

PROPOSITION X.

THEOREM.

Of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the same bears a greater Proportion, is the lesser Magnitude.

LET A have a greater Proportion to C, than B has to C. I say, A is greater than B.

For, if it be not greater, it will either be equal or less. But A is not equal to B, because
* 7 of this. then both A and B would have * the same Proportion to the Magnitude C; but they have not. Therefore A is not equal to B: Neither is it less than
† 8 of this. B; for then A would have † a less Proportion to C, than B would have; but it hath not a less Proportion: Therefore A is not less than B. But it has been proved likewise not to be equal to it: Therefore A shall be greater than B.

Again, let C have a greater Proportion to B than to A. I say, B is less than A.

For,

For, if it be not less, it is greater, or equal. Now, B is not equal to A, for then C would have * the same Proportion to A as to B; but this it has not. Therefore A is not equal to B; neither is B greater than A; for if it was, C would have † a less Proportion to B than to A; but it has not: Therefore B is not greater than A. But it has also been proved not to be equal to it. Wherefore B shall be less than A. Therefore, *of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the same bears a greater Proportion, is the lesser Magnitude*; which was to be demonstrated.

* 7 *of this.*

† 8 *of this.*

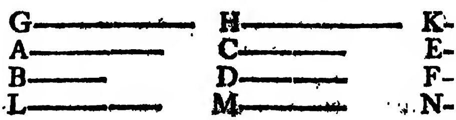
PROPOSITION XI.

Problem.

Proportions that are one and the same to any third, are also the same to one another.

LET A be to B, as C is to D; and C to D, as E to F. I say, A is to B, as E is to F.

For, take G, H, and K, Equimultiples of A, C, and



E; and L, M, and N, other Equimultiples of B, D, and F. Then, because A is to B, as C is to D, and there are taken G and H, the Equimultiples of A and C, and L and M, any other Equimultiples of B and D; if G exceeds L, * then H will exceed M; and if G be equal to L, H will be equal to M; and if less, less. Again, because as C is to D, so is E to F; and H and K are taken Equimultiples of C and E; as likewise M and N, any other Equimultiples of D and F; if H exceeds M *, then K will exceed N; and if H be equal to M, K will be equal to N; and if less, less. But if H exceeds M, G will also exceed L; if equal, equal; and if less, less. Wherefore, if G exceeds L, K will also exceed N; and if G be equal to L, K will be

* *Def.* 5 *of this.*

be equal to N; and if less, less. But G and K are Equimultiples of A and E; and L and N are Equimultiples of B and F. Consequently, as A is to B, so * is E to F. Therefore, *Proportions that are one and the same to any third, are also the same to one another*; which was to be demonstrated.

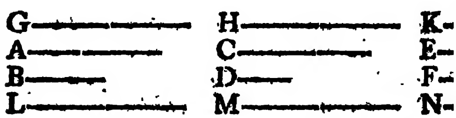
* Def. 5. of this.

PROPOSITION XII.

THEOREM.

If any Number of Magnitudes be proportional, as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents.

LET there be any Number of Proportional Magnitudes, A, B, C, D, E, F; whereof as A is to B,



so C is to D, and so E to F. I say, as A is to B, so are all the Antecedents A, C, and E, together, to all the Consequents, B, D, and F, together.

For, let G, H, and K, be Equimultiples of A, C, and E; and L, M, and N, any other Equimultiples of B, D, and F.

Then, because as A is to B, so is C to D, and so E to F; and G, H, and K, are Equimultiples of A, C, and E; and L, M, and N, Equimultiples of B, D, and F; if G exceeds L, H * will also exceed M, and K will exceed N; if G be equal to L, H will be equal to M, and K to N; and if less, less. Wherefore, also, if G exceeds L, then G, H, and K, together, will likewise exceed L, M, and N together; and, if G be equal to L, then G, H, and K, together, will be equal to L, M, and N, together; and if less, less: But G, and G, H, and K, are Equimultiples of A, and A, C, and E; because, if there are any Number of Magnitudes Equimultiples to a like Number of Magnitudes, each to the other, the same Multiple that one Magnitude is of one, so shall † all the Magnitudes be of all. And, for the

* Def. 5. of this.

† 1 of this.

the same Reason, L, and L, M, and N, are Equimultiples of B, and B, D, and F. Therefore, as A is to B, so * is A, C, and E, together, to B, D, and F, together. * Def. 5. of this.
Wherefore, *if there be any Number of Magnitudes proportional, as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents*; which was to be demonstrated.

PROPOSITION XIII.

Theorem.

If the first has the same Proportion to the second, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the sixth; then also shall the first have a greater Proportion to the second, than the fifth has to the sixth.

LET the first A have the same Proportion to the second B, as the third C has to the fourth D; and let the third C have a greater Proportion to the fourth D, than the fifth E to the sixth F. I say, likewise, that the

first A, to the second B, has a greater Proportion, than the fifth E, to the sixth F.

For, because C has a greater Proportion to D, than E has to F; there are * certain Equimultiples of C and E, and others of D and F, such that the Multiple of C may exceed the Multiple of D; but the Multiple of E not that of F. Now let these Equimultiples of C and E be G and H; and K and L those of D and F; so that G exceeds K, and H not L: Make M the same Multiple of A, as G is of C; and N the same of B, as K is of D. * Def. 7. of this.

Then, because A is to B, as C is to D; and M and G are Equimultiples of A and C; and N and K of B and D: If M exceeds N, then † G will exceed K; and if M be equal to N, G will be equal to K; and if less, † Def. 5.

less. But G does exceed K: Therefore M will also exceed N. But H does not exceed L. And M and H are Equimultiples of A and E; and N and L any others of B and F. Therefore A has a * greater Proportion to B, than E has to F. Wherefore, *if the first has the same Proportion to the second, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the sixth; then, also, shall the first have a greater Proportion to the second, than the fifth has to the sixth;* which was to be demonstrated.

* *Def.* 7. *of this.*

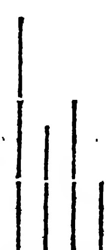
PROPOSITION XIV.

THEOREM.

If the first has the same Proportion to the second, as the third has to the fourth; and if the first be greater than the third; then will the second be greater than the fourth. But, if the first be equal to the third, then the second shall be equal to the fourth; and, if the first be less than the third, then the second will be less than the fourth.

LET the first A have the same Proportion to the second B, as the third C has to the fourth D; and let A be greater than C. I say, B is also greater than D.

8 *of this.* For, because A is greater than C, and B is any other Magnitude; A will have * greater Proportion to B, than C has to B: But as A is to B, so is C to D; there-
13 *of this.* fore, also, C shall ‡ have a greater Proportion to D, than C hath to B. But that Magnitude to which the same bears
10 *of this.* a greater Proportion, is † the lesser Magnitude: Wherefore D is less than B; and consequently B will be greater than D. In like manner we demonstrate, if A be equal to C, that B will be equal to D; and if A be less than C, that B will be less than D. Therefore, *if the first has the same Proportion to the second, as the third has to the fourth; and if the first be greater than the third; then will the second be greater than the fourth. But if the first be equal to the third,*


A B C D

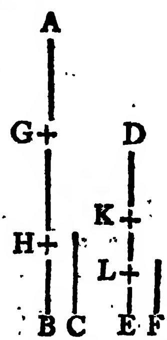
then the second shall be equal to the fourth; and if the first be less than the third, then the second will be less than the fourth; which was to be demonstrated.

PROPOSITION XV.

Theorem.

Parts have the same Proportion as their like Multiples, if taken correspondently.

LET AB be the same Multiple of C, as DE is of F. I say, as C is to F, so is AB to DE.



For, because AB and DE are Equimultiples of C and F, there shall be as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. Now, let AB be divided into the Magnitudes AG, GH, HB, each equal to C; and ED into the Magnitude DK, KL, LE, each equal to F; then the Number of the Magnitudes AG, GH, HB, will be equal to the Number of the Magnitudes DK, KL, LE. Now, because AG, GH, HB, are equal, as likewise DK, KL, LE; it shall be *, as AG is to DK, so is GH to KL, and so is HB to LE. But as one of the Antecedents is to one of the Consequents, so † all the Antecedents to all the Consequents. Therefore, as AG is to DK, so is AB to DE. But AG is equal to C, and DK to F. Whence, as C is to F, so shall AB be to DE. Therefore, *Parts have the same Proportion as their like Multiples, if taken correspondently;* which was to be demonstrated.

* 7 of this.

† 12 of this.

PRO-

PROPOSITION XVI.

TEHOREM.

If four Magnitudes of the same Kind are proportional, they shall also be alternately proportional.

LET four Magnitudes A, B, C, D, be proportional; whereof A is to B, as C is to D. I say, likewise, that they will be alternately proportional; *viz.* as A is to C, so is B to D: For take E and F, Equimultiples of A and B; and G and H, any Equimultiples of C and D

Then, because E is the same Multiple of A, as F is of B, and Parts have the same Pro-
* 15 of this. portion * to their like Multiples, if taken correspondently; it shall be, as A is to B, so is E to F. But as A is to B, so is C to D. Therefore, also, as C is to
† 11 of this. D, so † is E to F. Again, because G and H are Equimultiples of C and D, and Parts have the same Proportion with their like Multiples, if taken correspondently, it will be, as C is to D, so is G to H; but as C is to D, so is E to F. Therefore, also, as E is to F, so is G to H; and if four Magnitudes be proportional, and the first greater than the third, then the second
‡ 14 of this. will be ‡ greater than the fourth; and if the first be equal to the third, the second will be equal to the fourth; and if less, less. Therefore, if E exceeds G, F will exceed H; and if E be equal to G, F will be equal to H; and if less, less. But E and F are any Equimultiples of A and B; and G and H, any Equimultiples of C and D. Whence as A is to C, so shall
* Def. 5. B be * to D. Therefore, *if four Magnitudes of the same Kind are proportional, they shall also be alternately proportional*; which was to be demonstrated.

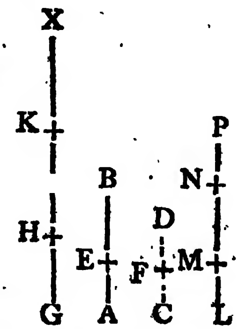
PRO-

PROPOSITION XVII.

Theorem.

If Magnitudes compounded are proportioned, they shall also be proportional when divided.

LET the compounded Magnitudes AB, BE, CD, DF, be proportional; that is, let AB be to BE, as CD is to DF. I say, these Magnitudes divided are proportional; *viz.* as AE is to EB, so is CF to FD.



For let GH, HK, LM, and MN, be Equimultiples of AE, EB, CF, and FD; and KX, and NP, any Equimultiples of EB and FD.

Because GH is the same Multiple of AE, as HK is of EB; therefore GH * is the same Multiple of AE, as GK is of AB. But GH is the same Multiple of AE, as LM is of CF. Wherefore GK is the same Multiple of AB, as LM is of CF. Again, because LM is the same Multiple of CF, as MN is of FD, LM will be * the same Multiple of CF, as LN is of CD. Therefore GK is the same Multiple of AB, as LN is of CD. And so GK and LN will be Equimultiples of AB and CD. Again, because HK is the same Multiple of EB, as MN is of FD; as likewise KX the same Multiple of EB, as NP is of FD; the compounded Magnitude HX is † also the same Multiple of EB, as MP is of FD. Wherefore, since it is, as AB is to BE, so is CD to DF; and GK and LN are Equimultiples of AB and CD; and also HX and MP any Equimultiples of EB and FD; if GK exceeds HX, then LN will ‡ exceed MP; and if GK be equal to HX, then LN will be equal to MP; if less, less. Now let GK exceed HX; then, if HK, which is common, be taken away, GH shall exceed KX. But when GK exceeds HX, then LN exceeds MP; therefore LN does exceed MP. If MN, which is common, be taken away, then

* 1 of this.

† 2 of this.

‡ Def. 5.

then LM will exceed NP. And so, if GH exceeds KX, then LM will exceed NP. In like manner we demonstrate, if GH be equal to KX, that LM will be equal to NP; and if less, less. But GH and LM are Equimultiples of AE and CF; and KX and NP are
* Def. 5. any Equimultiples of EB and FD. Whence, * as AE is to EB, so is CF to FD. Therefore, *if Magnitudes compounded are proportional, they shall also be proportional when divided*; which was to be demonstrated.

PROPOSITION XVIII.

Theorem.

If Magnitudes divided be proportional, the same also being compounded, shall be proportional.

LET the divided proportional Magnitudes be AE, EB, CF, FD; that is, as AE is to EB, so is CF to FD. I say, they are also proportional when compounded; *viz.* as AB is to BE, so is CD to DF.

For, if AB be not to BE, as CD is to DF, AB shall be to BE, as CD is to a Magnitude, either greater or less than FD.

First, Let it be to a lesser, *viz.* to GD. Then, because AB is to BE, as CD is to DG, compounded Magnitudes are pro-
* 17 of this. portional; and consequently * they will be proportional when divided. Therefore AE is to EB, as CG is to GD. But (by the *Hyp.*) as AE is to EB, so is CF to FD. Wherefore, also, as CG is to
† 11 of this. GD, so † is CF to FD. But the first CG is greater than the third CF; therefore the second DG shall be
‡ 14 of this. ‡ greater than the fourth DF. But it is less, which is absurd. Therefore AB is not to BE, as CD is to DG. We demonstrate in the same manner, that AB to BE is not as CD to a greater than DF. Therefore AB to BE must necessarily be as CD is to DF. And so, *if Magnitudes divided be proportional, they will also be proportional when compounded*; which was to be demonstrated.

PRO-

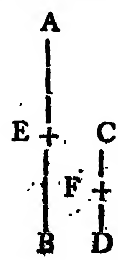
PROPOSITION XIX.

Theorem.

If the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole.

LET the Whole AB be to the Whole CD, as the Part taken away AE, is to the Part taken away CF. I say, the Residue EB is to the Residue FD, as the whole AB is to the Whole CD.

For, because the Whole AB is to the Whole CD, as AE is to CF; it shall be * alternately, as AB is to AE, so is CD to CF. Then, because compounded Magnitudes, being proportional, will be † also proportional when divided; as BE is to EA, so is DF to FC: And again, it will be by * Alternation, as BE is to DF, so is EA to FC. But as EA is to FC, so (by the *Hyp.*) is AB to CD. And therefore the ‡ Residue EB shall be to the Residue FD, as the Whole AB to the Whole CD. Wherefore, *if the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole*; which was to be demonstrated.



* 16 of this.

† 17 of this.

‡ 11 of this.

Coroll. If four Magnitudes be proportional, they will be likewise conversly proportional. For let AB be to BE, as CD to DF; then (by Alternation,) it shall be, as AB is to CD, so is BE to DF. Wherefore, since the Whole AB is to the Whole CD, as the Part taken away BE is to the Part taken away DF; the Residue AE to the Residue CF shall be as the Whole AB to the Whole CD. And again, (by Inversion and Alternation) as AB is to AE, so is CD to CF. Which is by converse Ratio.

The Demonstration of converse Ratio, laid down in this Corollary, is only particular; For Alternation (which is used herein) cannot be applied but when the four proportional Magnitudes are all of the same Kind, as will appear from the 4th and 17th Definitions of this Book. But converse Ratio may be used when the Terms of

of the first Ratio are not of the same Kind with the Terms of the latter. Therefore, instead of that, it may not be improper to add this Demonstration following: If four Magnitudes are proportional, they will be so conversly: For let AB *be to* BE, *as* CD *to* DF. *And then dividing, it is,* * *as* AE *is to* BE, *so is* CF *to* DF: *And this inversly is,* † *as* BE *is to* AE, *so is* DF *to* CF; *which by compounding becomes,* ‡ *as* AB *is to* AE, *so is* CD *to* CF; *which by the 17th Definition, is converse Ratio: By* S. Cunn.

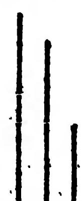
* 17 of this.
† Cor. 4. of this.
‡ 18 of this.

PROPOSITION XX.

Theorem.

If there be three Magnitudes, and others equal to them in Number, which, being taken two and two in each Order, are in the same Ratio; and if the first Magnitude be greater than the third, then the fourth will be greater than the sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth.

LET A, B, C, be three Magnitudes, and D, E, F, others equal to them in Number, which being taken two and two in each Order, are in the same Proportion, *viz.* let A be to B, as D is to E; and B to C, as E to F; and let the first Magnitude A be greater than the third C. I say, the fourth D is also greater than the sixth F. And if A be equal to C, D is equal to F. But if A be less than C, D is less than F.


A B C

For, because A is greater than C, and B is any other Magnitude; and since a greater Magnitude hath * a greater Proportion to the same Magnitude than a lesser hath; A will have a greater Proportion to B, than C hath to B. But as A is to B, so is D to E †; therefore D hath a greater Proportion to E, than C hath to B. Now inversly, as C is to B, so is F to E. Therefore also D will have a greater Proportion to E, than F has to E. But of Magnitudes having Proportion to the same Magnitude, that which has the greater

* 8 of this.
† By Hyp.


D E F

greater

greater Proportion is † the greater Magnitude. Therefore D is greater than F. In the same manner we demonstrate, if A be equal to C, then D will be also equal to F; and if A be less than C, then D will be less than F. Therefore, *if there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the same Ratio; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth*; which was to be demonstrated. † 10 of this.

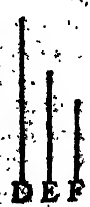
PROPOSITION XXI.

Theorem.

If there be three Magnitudes, and others equal to them in Number, which taken two and two, are in the same Proportion, and the Proportion be perturbate; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth; but if the first be equal to the third, then is the fourth equal to the sixth; if less, less.

LET three Magnitudes, A, B, C, be proportional; and others, D, E, F, equal to them in Number. Let their Analogy likewise be perturbate; *viz.* as A is to B, so is E to F; and as B is to C, so is D to E: If the first Magnitude A be greater than the third C, I say, the fourth D is also greater than the sixth F. And if A be equal to C, then D is equal to F; but if A be less than C, then D is less than F.

For, since A is greater than C, and B is some other Magnitude, A will have * a greater Proportion to B, than C has to B. But as A is to B, so is E to F; whence E has a greater Proportion to F, than C hath to B: Now inversly, as C is to B, so is E to D: Wherefore also, E shall have a greater Proportion to F, than E to D. But that Magnitude to which the same Magnitude * 8 of this.



bears

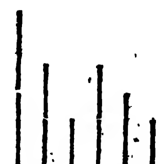
† 10 of this. bears a greater Proportion, † is the lesser Magnitude. Therefore F is less than D; and so D shall be greater than F. After the same manner we demonstrate, if A be equal to C, D will be also equal to F; and if A be less than C, D will also be less than F. *If, therefore, there are three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion, and the Proportion be perturbate; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth; but if the first be equal to the third, then is the fourth equal to the sixth; if less, less;* which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

If there be any Number of Magnitudes and others equal to them in Number, which, taken two and two, are in the same Proportion; then they shall be in the same Proportion by Equality.

LET there be any Number of Magnitudes, A, B, and C; and others, D, E, and F, equal to them in Number, which, taken two and two, are in the same Proportion; that is, as A is to B, so is D to E; and as B is to C, so is E to F. I say, they are also proportional by Equality; *viz.* as A is to C, so is D to F.

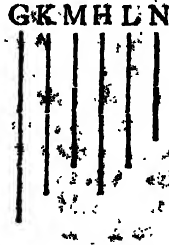


A B C D E F

For let G and H be any Equimultiples of A and D; and K and L any Equimultiples of B and E; and likewise M and N, any Equimultiples of C and F. Then, because A is to B, as D is to E; and G and H are Equimultiples of A and D, and K and L Equimultiples of B and E, it shall be, * as G is to K, so is H to L. For the same Reason also, it will be as K is to M, so is L to N. And since there are three Magnitudes G, K, and M, and others H, L, and N, equal to them in Number, which being taken two and two, in each Order,

* 4 of this.

G K M H L N



Order, are in the same Proportion; therefore if G exceeds M, * H will exceed N; if G be equal to M, then H shall be equal to N; and if G be less than M, H shall be less than N. But G and H are Equimultiples of A and D; and M and N, any other Equimultiples of C and F. Whence as A is to C, so shall D † be to F. Therefore, *if there be any Number of Magnitudes, and others equal to them in Number, which taken two and two, are in the same Proportion; then they shall be in the same Proportion by Equality*; which was to be demonstrated.

* 20 of this.

† Def. 5. of this.

PROPOSITION XXIII.

PROBLEM.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality.

LET there be three Magnitudes A, B, and C; and others equal to them in Number, D, E, and F, which, taken two and two, are in the same Proportion, and their Analogy be perturbate; that is, as A is to B, so is E to F; and as B is to C, so is D to E. I say, as A is to C, so is D to F.

For, let G, H, and L, be Equimultiples of A, B, and D; and K, M, and N, any Equimultiples of C, E, and F.

Then, because G and H are Equimultiples of A and B, and since Parts have the same Proportion as their like Multiples, when taken correspondently; it shall be *, as A is to B, so is G to H; and, by the same Reason, as E is to F, so is M to N. But A is to B, as E is to F. Therefore, ‡ as G is to H, so is M to N. Again, because B is to C as D is to E; and H and L are Equimultiples of B and D;

* 15 of this.

‡ 11 of this.

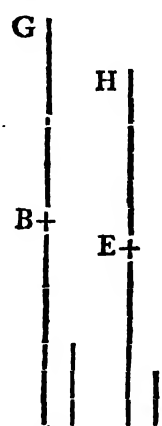
as likewise K and M any Equimultiples of C and F; it shall be, as H is to K, so is L to M. But it has been also proved, that as G is to H, so is M to N. Therefore, because three Magnitudes G, H, and K, and others, L, M, and N, equal to them in Number, which, taken two and two, are in the same Proportion, and their Analogy is perturbate; then if G exceeds K, also L * will exceed N; and G be equal to K, then L will be equal to N; and if G be less than K, L will likewise be less than N. But G and L are Equimultiples of A and D; and K and N Equimultiples of C and F. Therefore, as A is to C, so shall D be to F. Wherefore, *if there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality*; which was to be demonstrated.

* 21 of this.

PROPOSITION XXIV.

THEOREM.

If the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second, as the sixth has to the fourth; then shall the first compounded with the fifth, have the same Proportion to the second, as the third, compounded with the sixth, has to the fourth.



LET the first Magnitude AB have the same Proportion to the second C, as the third DE has to the fourth F. Let also the fifth BG have the same Proportion to the second C, as the sixth EH has to the fourth F. I say, AG, the first compounded with the fifth has the same Proportion to the second C, as DH, the third compounded with the sixth, has to the fourth F.

For,

For, because BG is to C, as EH is to F; it shall be (inversly), as C is to BG, so is F to EH. Then, since AB is to C, as DE is to F; and as C is to BG, so is F to EH; it shall be, * by Equality, as AB is to BG, so is DE to EH. And because Magnitudes, being divided, are proportional, they shall also be † proportional when compounded. Therefore, as AG is to GB, so is DH to HE: But as GB is ‡ to C, so also is HE to F. Wherefore, by Equality *, it shall be, as AG is to C, so is DH to F. Therefore, *if the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second, as the sixth has to the fourth; then shall the first, compounded with the fifth, have the same Proportion to the second, as the third compounded with the sixth, has to the fourth*; which was to be demonstrated.

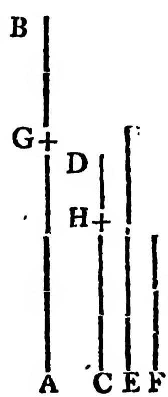
* 22 of this.
† 18 of this.
‡ Hyp.

PROPOSITION XXV.

THEOREM.

If four Magnitudes be proportional; the greatest, and the least of them, will be greater than the other two.

LET four Magnitudes, AB, CD, E, F, be proportional, whereof AB is to CD, as E is to F; let AB be the greatest of them, and F the least. I say, AB and F, are greater than CD and E.



For, let AG be equal to E, and CH to F. Then, because AB is to CD as E is to F; and since AG and CH are each equal to E and F; it shall be as AB is to DC, so is AG to CH. And because the Whole AB is to the Whole CD, as the Part taken away AG is to the Part taken away CH; it shall also be *, as the Residue GB to the Residue HD, so is the Whole AB to the Whole CD. But AB is greater than CD; therefore, also, GB shall be greater than HD. And since AG is equal to E,

* 29 of this.

and CH to F; AG and F will be equal to CH and E. But if equal Things are added to unequal Things, the Wholes ſhall be unequal. Therefore GB, HD, being unequal, for GB is the greater, if AG and F are added to GB; and CH and E to HD; then AB and F will neceſſarily be greater than CD and E. Wherefore, *if four Magnitudes be proportional; the greateſt, and the leaſt of them will be greater than the other two;* which was to be demonſtrated.

The END *of the* FIFTH BOOK.

EUCLID's

ELEMENTS.

BOOK VI.

DEFINITIONS.

I. *SIMILAR Right-lined Figures are such as have each of their several Angles equal to one another, and also the Sides about the equal Angles proportional to each other.*

II. *Figures are said to be reciprocal, when the antecedent and consequent Terms of the Ratios* [illegible]

III. [illegible] *said to be cut into mean and extreme Proportion, when the Whole is to the greater Segment, as the greater Segment is to the lesser.*

IV. *The Altitude of any Figure is a perpendicular Line drawn from the Top, or Vertex, to the Base.*

V. *A Ratio is said to be compounded of Ratios, when the Quantities of the Ratios, being multiplied into one another, do produce a Ratio.*

PROPOSITION I.

THEOREM.

Triangles and Parallelograms, that have the same Altitude, are to each other as their Bases.

LET the Triangles ABC, ACD, and the Parallograms EC, CF, have the same Altitude, *viz.* the Perpendicular drawn from the Point A to BD. I say, as the Base BC is to the Base CD, so is the Triangle ABC, to the Triangle ACD; and so is the Parallelogram EC to the Parallelogram CF.

For, produce BD both Ways to the Points H and L; and take GB, GH, any Number of Times equal to the Base BC; and DK, KL, any Number of Times equal to the Base CD; and join AH, AG, AK, AL.

Then, because CB, BG, GH, are equal to one another, the Triangles AHG, AGB, ABC, also, will be * equal to one another: Therefore the same Multiple that the Base HC is of the Base BC, shall the Triangle AHC be of the Triangle ABC. By the same Reason, the same Multiple that the Base LC is of the Base CD, shall the Triangle ALC be of the Triangle ACD. And if the Base HC be equal to the Base CL, the Triangle AHC is also * equal to the Triangle ALC: And if the Base HC exceeds the Base CL, then the Triangle AHC will exceed the Triangle ALC. And if the Base HC be less than CL, then the Triangle AHC will be less than ALC. Therefore, since there are four Magnitudes, *viz.* the two Bases BC, CD, and the two Triangles ABC, ACD; and since the Base HC, and the Triangle AHC, are Equimultiples of the Base BC, and the Triangle ABC: And the Base CL, and the Triangle ALC, are Equimultiples of the Base CD, and the Triangle ADC: And it has been proved, that if the Base HC exceeds the Base CL, the Triangle AHC will exceed the Triangle ALC; and if equal, equal; if less, less: Therefore, as the Base BC is to the Base CD, so † is the Triangle ABC to the Triangle ACD.

* 38. 1.

† Def. 5. 5.

And

And because the Parallelogram EC is † double to the Triangle ABC; and the Parallelogram FC, double † to the Triangle ACD; and Parts have the same Proportion * as their like Multiples: Therefore, as the Triangle ABC is to the Triangle ACD, so is the Parallelogram EC to the Parallelogram CF. And so, since it has been proved that the Base BC is to the Base CD, as the Triangle ABC is to the Triangle ACD; and the Triangle ABC is to the Triangle ACD, as the Parallelogram EC is to the Parallelogram CF; it shall be ‡, as the Base BC is to the Base CD, so is the Parallelogram EC to the Parallelogram FC. Wherefore, *Triangles and Parallelograms, that have the same Altitude, are to each other as their Bases*; which was to be demonstrated. † 41. 1. * 15. 1. ‡ 11. 5.

PROPOSITION II.

THEOREM.

If a Right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line, joining the Points of Section, shall be parallel to the other Side of the Triangle.

LET DE be drawn parallel to BC, a Side of the Triangle ABC. I say, DB is to DA, as CE is to EA.

For, let BE, CD be joined.

Then the Triangle BDE is * equal to the Triangle CDE; for they stand upon the same Base DE, and are between the same Parallels DE and BC; and ADE is some other Triangle. But equal Magnitudes have † the same Proportion to one and the same Magnitude. Therefore, as the Triangle BDE is to the Triangle ADE, so is the Triangle CDE to the Triangle ADE. * 37. 1. † 7. 5.

But as the Triangle BDE is to the Triangle ADE, so ‡ is BD to DA; for since they have the same Altitude, *viz.* a Perpendicular drawn from the Point E to AB, they are to each other as their Bases. And, for the same Reason, as the Triangle CDE is to the Tri- ‡ 1 of this.

angle ADE, so is CE to EA: And therefore as BD
* 11. 5. is to DA, so is * CE to EA.

And if the Sides AB, AC, of the Triangle ABC, be cut proportionally; that is, so that BD be to DA, as CE is to EA; and if DE be joined; I say, DE is parallel to BC.

For, the same Construction remaining, because BD
† 1 of this. is to DA, as CE is to EA; and BD is † to DA, as the Triangle BDE is to the Triangle ADE, and CE is to EA, as the Triangle CDE is to the Triangle ADE; it shall be as the Triangle BDE is to the Triangle ADE, so is * the Triangle CDE to the Triangle ADE. And since the Triangle BDE, CDE, have the same Proportion to the Triangle ADE, the Tri-
† 9. 5. angle BDE shall be † equal to the Triangle CDE; and they have the same Base DE: But equal Tri-
‡ 39. 1. angles, being upon the same Base, ‡ are between the same Parallels; therefore DE is parallel to BC. Wherefore, *if a Right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line, joining the Points of Section, shall be parallel to the other Side of the Triangle*; which was to be demonstrated.

PROPOSITION III.

THEOREM.

If one Angle of a Triangle be bisected, and the Right Line, that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a Right Line, drawn from the Vertex, to the Point of Section of the Base, will bisect the Angle of the Triangle.

LET there be a Triangle ABC, and let its Angle
* 9. 1. BAC be * bisected by the Right Line AD, I say, as BD is to DC, so is BA to AC.

For thro' C draw * CE parallel to DA, and produce BA, till it meets CE in the Point E. * 31. 1.

Then, because the Right Line AC falls on the Parallels AD, EC, the Angle ACE will be ‡ equal to the Angle CAD: But the Angle CAD (by the *Hypothesis*) is equal to the Angle BAD. Therefore the Angle BAD will be equal to the Angle ACE. Again, because the Right Line BAE falls on the Parallels AD, EC, the outward Angle BAD is ‡ equal to the inward Angle AEC; but the Angle [redacted] has been proved equal to the Angle BAD: Therefore ACE shall be equal to AEC; and so the Side AE is equal ‡ to the Side AC. And because the Line AD is drawn parallel to CE, the Side of the Triangle BCE, it shall be, * as BD is to DC, so is BA to AE; but AE is equal to AC. Therefore, as BD is to DC, so is † BA to AC. ‡ 29. 1. [redacted] ACE. ‡ 6. 1. * 2 of this.

And if BD be to DC, as BA is to AC; and the Right Line AD be joined; then, I say, the Angle BAC is bisected by the Right Line AD.

For, the same Construction remaining, because BD is to DC, as BA is to AC; and as BD is to DC, so is ‡ BA to AE; for AD is drawn parallel to one Side EC of the Triangle BCE; it shall be, as BA is to AC, so is BA to AE. Therefore AC is equal to AE †; and, accordingly, the Angle AEC is equal to the Angle ECA: But the Angle AEC is equal * to the outward Angle BAD; and the Angle ACE equal to the alternate Angle CAD. Wherefore the Angle BAD is also equal to the Angle CAD; and so the Angle BAC is bisected by the Right Line AD. Therefore, *if one Angle of a Triangle be bisected, and the Right Line, that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a Right Line, drawn from the Vertex, to the Point of Section of the Base, will bisect the Angle of the Triangle*; which was to be demonstrated. ‡ 2 of this. † 9. 4. * 29. 1.

PRO-

PROPOSITION IV.

THEOREM.

The Sides about the equal Angles of equiangular Triangles are proportional; and the Sides, which are ſubtended under the equal Angles, are homologous, or of like Ratio.

LET ABC, DEC, be equiangular Triangles, having the Angle ABC equal to the Angle DCE, the Angle ACB equal to the Angle DEC, and the Angle BAC equal to the Angle CDE. I ſay, the Sides that are about the equal Angles of the Triangles ABC, DCE, are proportional; and the Sides that are ſubtended under the equal Angles are homologous, or of like Ratio.

Set the Side BC in the ſame Right Line with the Side CE; and becauſe the Angles ABC, ACB, are
* 17. 1. * leſs than two Right Angles, and the Angle ACB is equal to the Angle DEC, the Angles ABC, DEC, are leſs than two Right Angles. And ſo BA, ED,
† Ax. 12. produced, will meet * each other; let them be produced, and meet in the Point F. Then, becauſe the Angle DCE is equal to the Angle ABC, BF ſhall be
‡ 23. 1. ‡ parallel to DC. Again, becauſe the Angle ACB is equal to the Angle DEC, the Side AC will be ‡ parallel to the Side FE; therefore FACD is a Parallelo-
* 34. 1. gram, and conſequently FA is * equal to DC, and AC to FD; and becauſe AC is drawn parallel to FE, the
† 2 of this. Side of the Triangle FBE, it ſhall † be, as BA is to AF, ſo is BC to CE: But CD is equal to AF, and (by Alternation) as BA is to BC, ſo is CD to CE. Again, becauſe CD is parallel to BF, it ſhall be, † as BC is to CE, ſo is FD to DE, but FD is equal to
‡ 7. 5. AC. Therefore as BC is to CE, ſo is ‡ AC to DE: And ſo by Alternation, as BC is to CA, ſo is CE to ED. Wherefore, becauſe it is demonſtrated, that AB is to BC, as DC is to CE; and as BC is to CA, ſo is
* 21. 5. CE to ED; it ſhall be, * by Equality, as BA is to AC, ſo is CD to DE. Therefore, *the Sides about the equal Angles of equiangular Triangles are proportional; and the Sides, which are ſubtended under the equal Angles,*

are

are homologous, or of like Ratio; which was to be demonstrated.

PROPOSITION V.

THEOREM.

If the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologous Sides are subtended, are equal.

LET there be two Triangles ABC, DEF, having their Sides proportional; that is, let AB be to BC, as DE is to EF; and as BC to CA, so is EF to FD: And, also, as BA to CA, so ED to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angles are equal, under which the homologous Sides are subtended; *viz.* the Angle ABC equal to the Angle DEF; and the Angle BCA equal to the Angle EFD; and the Angle BAC equal to the Angle EDF.

For, at the Points E and F, with the Line EF, make * the Angle FEG equal to the Angle ABC; and the Angle EFG equal to the Angle BCA: Then the remaining Angle BAC is † equal to the remaining Angle EGF. * 23. 1. † *Cor.* 32. 1.

And so the Triangle ABC is equiangular to the Triangle EGF; and, consequently, the Sides that are subtended under the equal Angles, are proportional. Therefore, as AB is to BC, so is ‡ GE to EF; but (by the *Hyp.*) as AB is to BC, so is DE to EF: Therefore, as DE is to EF, so is * GE to EF. And since DE, EG, have the same Proportion to EF, DE shall be † equal to EG. For the same Reason DF is equal to FG; but EF is common. Then, because the two Sides DE, EF, are equal to the two Sides GE, EF, and the Base DF is equal to the Base FG, the Angle DEF is ‡ equal to the Angle GEF; and the Triangle DEF equal to the Triangle GEF; and the other Angles of the one equal to the other Angles of the other, which are subtended by the equal Sides. Therefore the Angle DFE is equal to the Angle GFE, and the Angle EDF equal to the Angle EGF. And because the ‡ 4 *of this.* * 11. 5. † 9. 5. ‡ 8. 1.

the Angle DEF is equal to the Angle GEF; and the Angle GEF equal to the Angle ABC; therefore the Angle ABC shall be also equal to the Angle DEF. For the same Reason, the Angle ACB shall be equal to the Angle DFE; as also the Angle A equal to the Angle D: Therefore the Triangle ABC will be equiangular to the Triangle DEF. Wherefore, *if the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologous Sides are subtended, are equal*; which was to be demonstrated.

PROPOSITION VI.

THEOREM.

If two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular; and have those Angles equal, under which are subtended the homologous Sides.

LET there be two Triangles ABC, DEF, having one Angle BAC, of the one, equal to the Angle EDF of the other; and let the Sides about the equal Angles be proportional; *viz.* let AB be to AC, as ED is to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC equal to the Angle DEF; and the Angle ACB equal to the Angle DFE.

For, at the Points D and F, with the Right Line DF, make * the Angle FDG equal to either of the Angles BAC, EDF; and the Angle DFG equal to the Angle ACB.

* 23. 1.

Then the other Angle B is † equal to the other Angle G; and so the Triangle ABC is equiangular to the Triangle DGF; and, consequently, as BA is to AC, so is ‡ GD to DF: But (by the *Hyp.*) as BA is to AC, so is ED to DF. Therefore, as ED is * to DF, so is GD to DF; whence ED is † equal to DG, and DF is common; therefore the two Sides ED, DF, are equal to the two Sides GD, DF; and the Angle EDF equal to the Angle GDF: Consequently the

† Cor. 32. 1.
‡ 4 of this.
* 11. 5.
† 9. 5.

the Base EF is * equal to the Base FG, and the Triangle DEF equal to the Triangle DGF; and the other Angles of the one equal to the other Angles of the o[illegible], each to each; under which the equal Sides are subtended. Therefore the Angle DFG is equal to the Angle DFE, and the Angle G equal to the Angle E; but the Angle DFG is equal to the Angle ACB: Wherefore the Angle ACB is equal to the Angle DFE; but the Angle BAC is ‡ also equal to the Angle EDF: Therefore the other Angle at B is † equal to the other Angle at E; and so the Triangle ABC is equiangular to the Triangle DEF. Therefore, *if two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional; then the Triangles are equiangular; and have those Angles equal, under which are subtended the homologous Sides;* which was to be demonstrated.

* 4.
‡ By Hyp.
† 32. 1.

PROPOSITION VII.

THEOREM.

If there are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about the other Angles proportional; and if the remaining third Angles are either both less, or both not less, than Right Angles; then shall the Triangles be equiangular, and have those Angles equal, about which are the proportional Sides.

LET two Triangles ABC, DEF, have one Angle of the one equal to one Angle of the other, *viz.* the Angle BAC equal to the Angle EDF; and let the Sides about the other Angles ABC, DEF, be proportional; *viz.* as DE is to EF, so let AB be to BC; and let the other Angles at C and F be both less, or both not less, than Right Angles. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC is equal to the Angle DEF; as also the other Angle at C equal to the other Angle at F.

For, if the Angle ABC be not equal to the Angle DEF, one of them will be the greater, which let be ABC. Then at the Point B with the Right Line

AB,

* 23. 1. AB, make * the Angle ABG equal to the Angle DEF.

Now, because the Angle A is equal to the Angle D, and the Angle ABG equal to the Angle DEF;
† Cor. 32. 1. the remaining Angle AGB is † equal to the remaining Angle DFE: And therefore the Triangle ABG is equiangular to the Triangle DEF; and so, as AB
‡ 4 of this. is to BG, so is ‡ DE to EF; but as DE is to EF, so
* By Hyp. is * AB to BC. Therefore, as AB is to BC, so is
‡ 11. 5. AC to ‡ BG; and since AB has the same Proportion
† 9. 5. to BC, that is, as to BG, BC shall be † equal to BG;
* 5. 1. and, consequently, the Angle at C * equal to the Angle BGC. Wherefore each of the Angles BCG, or BGC, is less than a Right Angle; and consequently AGB is greater than a Right Angle. But the Angle AGB has been proved equal to the Angle at F; therefore the Angle at F is greater than a Right Angle: But (by the *Hyp.*) it is not greater, since C is not greater than a Right Angle, which is absurd. Wherefore the Angle ABC is not unequal to the Angle DEF; and so it must be equal to the same; but the Angle at A is equal to that at D; wherefore the Angle remaining at C is equal to the remaining Angle at F; and, consequently, the Triangle ABC is equiangular to the Triangle DEF. Therefore, *if there are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about the other Angles proportional; and if the remaining third Angles are either both less, or both not less, than Right Angles; then shall the Triangles be equiangular, and have those Angles equal, about which are the proportional Sides*; which was to be demonstrated.

PRO-

PROPOSITION VIII.

THEOREM.

If a Perpendicular be drawn, in a Right-angled Triangle, from the Right Angle to the Base, then the Triangle on each Side of the Perpendicular are similar, both to the Whole, and also to one another.

LET ABC be a Right-angled Triangle, whose Right Angle is BAC; and let the Perpendicular AD, be drawn from the Point A to the Base BC. I say, the Triangles ABD, ADC, are similar to one another, and to the whole Triangle ABC.

For, because the Angle BAC is equal to the Angle ADB, for each of them is a Right Angle; and the Angle at B is common to the two Triangles ABC, ABD; the remaining Angle ACB shall be * equal to the remaining Angle BAD. Therefore the Triangle ABC is equiangular to the Triangle ABD; and so, as † BC, which subtends the Right Angle of the Triangle ABC, is to BA, subtending the Right Angle of the Triangle ABD, so is AB, subtending the Angle C of the Triangle ABC, to DB, subtending an Angle equal to the Angle C; *viz.* the Angle BAD, of the Triangle ABD; and so, moreover, is AC to AB, subtending the Angle B, which is common to the two Triangles. Therefore the Triangle ABC is ‡ equiangular to the Triangle ABD; and the Sides about the equal Angles are proportional. Wherefore the Triangle ABC is ‡ similar to the Triangle ABD. By the same Way we demonstrate, that the Triangle ADC is also similar to the Triangle ABC. Wherefore each of the Triangles ABD, ABC, is similar to the whole Triangle.

* Cor. 32. 1.
† 4 of this.
‡ Def. 1. of this.

I say, the said Triangles are also similar to one another.

For, because the Right Angle BDA is equal to the Right Angle ADC, and the Triangle BAD has been proved equal to the Angle C; it follows, that the remaining Angle at B * shall be equal to the remaining * Cor. 32. 1.

Angle

Angle DAC. And so the Triangle ABD is equiangular to the Triangle ADC. Wherefore as † BD, subtending the Angle BAD of the Triangle ABD, is to DA, subtending the Angle at C of the Triangle ADC, which is equal to the Angle BAD; so is AD, subtending the Angle B of the Triangle ABD, to DC, subtending the Angle DAC, equal to the Angle B. And, moreover, so is BA to AC, subtending the Right Angles at D; and, consequently, the Triangle ABD is similar to the Triangle ADC. Wherefore, *if a Perpendicular be drawn in a Right-angled Triangle, from the Right Angle to the Base, then the Triangles on each Side of the Perpendicular are similar, both to the Whole, and also to one another*; which was to be demonstrated.

† 4 *of this.*

Coroll. From hence it is manifest, that the Perpendicular drawn in a Right-angled Triangle from the Right Angle to the Base, is a mean Proportional between the Segments of the Base. Moreover, either of the Sides containing a Right Angle, is a mean Proportional between the whole Base, and the Segment thereof, which is next to the Side.

PROPOSITION IX.

Problem.

To cut off any Part required from a given Right Line.

LET AB be a Right Line given, from which must be cut off any required Part; suppose a third.

Draw any Right Line AC from the Point A, making an Angle at Pleasure with the Line AB. Assume any Part D in the Line AC; make * DE, EC, each equal to AD; join BC, and draw † DF thro' D, parallel to BC.

* 3. 1.
† 31. 1.

Then, because FD is drawn parallel to the Side BC of the Triangle ABC, it shall be, ‡ as CD is to DA, so is BF to FA. But CD is double to DA. Therefore BF shall be double to FA; and so BA is triple to AF. Wherefore, *there is cut off* AF, *a third Part required, of the given Right Line* AB; which was to be done.

‡ 2 *of this.*

PRO-

PROPOSITION X.

PROBLEM.

To divide a given undivided Right Line, as another given Right Line is divided.

LET AB be a given undivided Right Line, and AC a divided Line. It is required to divide AB, as AC is divided.

Let AC be divided in the Points D and E, and so placed, as to contain any Angle with AB. Join the Points C and B; through D and E let DF, EG, be drawn * parallel to BC; and through D draw DHK, parallel to AB. * 31. 1.

Then FH, HB, are each of them Parallelograms; and so DH is † equal to FG, and HK to GB. And because HE is drawn parallel to the Side KC of the Triangle DKC, it shall be ‡ as CE is to ED, so is KH to HD. But KH is equal to BG, and HD to GF. Therefore, as CE is to ED, so is BG to GF. Again, because FD is drawn parallel to the Side EG of the Triangle AGE, as ED is to DA, so shall ‡ GF be to FA. But it has been proved, that CE is to ED, as BG is to GF. Therefore, as CE is to ED, so is BG to GF; and as ED is to DA, so is GF to FA. Wherefore, *the given undivided Line* AB *is divided as the given Line* AC *is*; which was to be done. † 34. 1. ‡ 2 *of this.*

PROPOSITION XI.

PROBLEM.

Two Right Lines being given, to find a third Proportional to them.

LET AB, AC, be two given Right Lines, so placed, as to make any Angle with each other. It is required to find a third Proportional to AB, AC.

Produce AB, AC, to the Points D and E; make BD equal to AC; join the Points B, C; and draw ‡ the Right Line DE thro' D, parallel to BC. ‡ 31. 1.

Then, becauſe BC is drawn parallel to the Side DE
‡ 2 of this. of the Triangle ADE, it ſhall be, ‡ as AB is to BD, ſo is AC to CE. But BD is equal to AC. Hence, as AB is to AC, ſo is AC to CE. Therefore, *a third Proportional* CE *is found to two given Right Lines* AB, AC; which was to be done.

PROPOSITION XII.

PROBLEM.

Three Right Lines being given, to find a fourth Proportional to them.

LET A, B, C, be three Right Lines given. It is required to find a fourth Proportional to them.

Let DE and DF be two Right Lines, making any Angle EDF with each other. Now make DG equal to A, GE equal to B, DH equal to C; and draw the
† 31. 3. Line GH, as alſo, † EF thro' E, parallel to GH.

Then, becauſe GH is drawn parallel to EF, the Side of the Triangle DEF, it ſhall be, as DG is to GE, ſo is DH to HF. But DG is equal to A, GE to B, and DH to C. Conſequently, as A is to B, ſo is C to HF. Therefore, *the Right Line* HF, *a fourth Proportional to the three given Right Lines* A, B, C, *is found*; which was to be done.

PROPOSITION XIII.

PROBLEM.

To find a mean Proportional between two given Right Lines.

LET the two given Right Lines be AB, BC. It is required to find a mean Proportional between them. Place AB, BC, in a direct Line; and on the
‡ 11. 1. Whole AC deſcribe the Semicircle ADC, and ‡ draw BD at Right Angles to AC from the Point B; and let AD, DC, be joined.

Then, becauſe the Angle ADC, in a Semicircle,
† 31. 3. is † a Right Angle; and ſince the Perpendicular DB is drawn from the Right Angle to the Baſe; therefore

DB

DB is a mean Proportional between the Segments of the Base AB, BC. Wherefore, *a mean Proportional between the two given Lines* AB, BC, *is found*; which was to be done. ‡ *Cor.* 8. *of this.*

PROPOSITION XIV.

THEOREM.

Equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal.

LET AB, BC, be equal Parallelograms, having the Angles at B equal, and let the Sides DB, BE, be in one strait Line; then also will ‡ the Sides FB, BG, be in one strait Line. I say, the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocal; that is, as DB is to BE, so is GB to BF. ‡ 14. 1.

For, let the Parallelogram FE be compleated.

Then, because the Parallelogram AB is equal to the Parallelogram BC, and FE is some other Parallelogram; it shall be, as AB is to FE, so is † BC to FE; but as AB is to FE, so is ‡ DB to BE; and as BC is to FE, so is GB to BF. Therefore as DB is to BE, so is GB to BF. Wherefore the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocally proportional. † 7. 5. ‡ 1 *of this.*

And if the Sides that are about the equal Angles are reciprocally proportional; *viz.* if DB be to BE, as GB is to BF; I say, the Parallelogram AB is equal to the Parallelogram BC.

For, since DB is to BE, as GB is to BF; and DB to BE, as the Parallelogram AB ‡ to the Parallelogram FE; and GB ‡ to BF, as the Parallelogram BC to the Parallelogram FE; it shall be as AB is to FE, so is BC † to FE. Therefore the Parallelogram AB is equal to the Parallelogram BC. And so, *equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles*

 gles

gles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal; which was to be demonstrated.

PROPOSITION XV.

THEOREM.

Equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and those Triangles that have one Angle of the one equal to one Angle of the other, and have also the Sides about the equal Angles reciprocal, are equal.

LET the equal Triangles ABC, ADE, have one Angle of the one equal to one Angle of the other; *viz.* the Angle BAC equal to the Angle DAE. I say, the Sides about the equal Angles are reciprocal; that is, as CA is to AD, so is EA to AB.

For, place CA and AD in one strait Line; then
‡ 14. 1. EA and AB shall be ‡ also in one strait Line; and let BD be joined. Then, because the Triangle ABC is equal to the Triangle ADE, and ABD is some other
† 7. 5. Triangle, the Triangle CAB shall be † to the Triangle BAD, as the Triangle ADE is to the Triangle BAD. But, as the Triangle CAB is to the Triangle
‡ 1 *of this.* BAD, so is CA ‡ to AD; and as the Triangle EAD is to the Triangle BAD, so ‡ is EA to AB. There-
* 11. 5. fore, as CA is to AD, * so is EA to AB. Wherefore the Sides of the Triangles ABC, ADE, about the equal Angles are reciprocal.

And, if the Sides about the equal Angles of the Triangles ABC, ADE, be reciprocal, *viz.* if CA be to AD, as EA is to AB; I say, the Triangle ABC is equal to the Triangle ADE.

For, again, let BD be joined. Then, because CA is to AD, as EA is to AB; and CA to AD ‡, as the Triangle ABC to the Triangle BAD; and EA to AB ‡, as the Triangle EAD to the Triangle BAD; therefore, as the Triangle ABC is to the Triangle BAD, * so shall the Triangle EAD be to the Triangle BAD.

BAD. Whence the Triangles ABC, ADE, have the ſame Proportion to the Triangle BAD; and ſo the Triangle ABC is ‡ equal to the Triangle ADE. ‡ 7. 5.
Therefore, *equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and thoſe Triangles that have one Angle of the one equal to one Angle of the other, and have alſo the Sides about the equal Angles reciprocal, are equal*; which was to be demonſtrated.

PROPOSITION XVI.

THEOREM.

If four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional.

LET four Right Lines AB, CD, E, F, be proportional, ſo that AB be to CD, as E is to F. I ſay, the Rectangle contained under the Right Lines AB and F, is equal to the Rectangle contained under the Right Lines CD and E.

For, draw AG and CH, from the Points A and C, at Right Angles to AB and CD; and make AG equal to F, and CH equal to E; and let the Parallelograms BG, DH, be compleated.

Then, becauſe AB is to CD, as E is to F; and ſince CH is equal to E, and AG to F; it ſhall be, as AB is to CD, ſo is CH to AG. Therefore, the Sides that are about the equal Angles of the Parallelograms BG, DH, are reciprocal; and ſince thoſe Parallelograms are equal *, that have the Sides about the equal Angles reciprocal; therefore the Parallogram BG is equal to the Parallelogram DH. But the Parallelogram BG is equal to that contained under AB and F; for AG is equal to F, and the Parallelogram DH equal to that contained under CD and E, ſince CH is equal to E. Therefore the Rectangle contained under AB and F, is equal to that contained under CD and E. * 14 *of this.*

And if the Rectangle contained under AB and F be equal to the Rectangle contained under CD and E; I say, the four Right Lines are Proportionals; *viz.* as AB is to CD, so is E to F.

For, the same Construction remaining, the Rectangle contained under AB and F is equal to that contained under CD and E; but the Rectangle contained under AB and F is the Rectangle BG; for AG is equal to F; and the Rectangle contained under CD and E is the Rectangle DH; for CH is equal to E. Therefore the Parallelogram BG shall be equal to the Parallelogram DH, and they are equiangular; but the Sides of equal and equiangular Parallelograms, which are
* 14 *of this.* about the equal Angles, are * reciprocal. Wherefore, as AB is to CD, so is CH to AG; but CH is equal to E, and AG to F; therefore, as AB is to CD, so is E to F. Wherefore, *if four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional*; which was to be demonstrated.

PROPOSITION XVII.

THEOREM.

If three Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three Right Lines are proportional.

LET there be three Right Lines, A, B, C, proportional; and let A be to B, as B is to C. I say, the Rectangle, contained under A and C, is equal to the Square of B.

For, make D equal to B.

Then, because A is to B as B is to C; and B is
* 7. 5. equal to D; it shall be *, as A is to B so is D to C. But, if four Right Lines be Proportionals, the Rectan-
† 16 *of this.* gle contained under the Extremes is † equal to the Rectangle under the Means. Therefore the Rectan-

gle

gle contained under A and C is equal to the Rectangle under B and D: But the Rectangle under B and D is equal to the Square of D; for B is equal to D: Wherefore the Rectangle contained under A, C, is equal to the Square of B.

And if the Rectangle contained under A and C be equal to the Square of B: I say, as A is to B, so is B to C.

For, the same Construction remaining, the Rectangle contained under A and C, is equal to the Square of B; but the Square of B is the Rectangle contained under B and D; for B is equal to D; and the Rectangle contained under A and C shall be equal to the Rectangle contained under B and D. But if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, the four Right Lines shall be † Proportionals. Therefore A is to B, as D is to C; but B is equal to D. Wherefore A is to B, as B is to C. Therefore, *if three Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three Right Lines are proportional*; which was to be demonstrated. † 16 *of this.*

PROPOSITION XVIII.

PROBLEM.

Upon a given Right Line, to describe a Right lined Figure, similar, and similarly situate, to a Right-lined Figure given.

LET AB be the Right Line given, and CE the Right-lined Figure. It is required to describe upon the Right Line AB a Figure similar, and similarly situate, to the Right-lined Figure CE.

Join DF, and make, * at the Points A and B, with the Line AB, the Angles GAB, ABG, severally equal to the Angles C and CDF. Whence the other Angle CFD is † equal to the other Angle AGB; and so the Triangle FCD is equiangular to the Triangle GAB: And, consequently, as FD is to GB, so is ‡ FC to GA; and so is CD to AB. Again, make the * 23. 1. † *Cor.* 32. 1. ‡ 4 *of this.*

the Angles BGH, GBH, at the Points B and G, with the Right Line BG, severally equal to the Angles

† Cor. 32. 1. EFD, EDF; then the remaining Angle at E is † equal to the remaining Angle at H. Therefore the Triangle FDE is equiangular to the Triangle GBH; and, con-

‡ 4 of this. sequently, as FD is to GB, so is ‡ FE to GH; and so ED to HB. But it has been proved, that FD is to GB, as FC is to GA, and as CD to AB. And therefore, as

* 11. 5. FC is to AG, so is * CD to AB; and so FE to GH; and so ED to HB. And because the Angle CFD is equal to the Angle AGB; and the Angle DFE equal to the Angle BGH; the whole Angle CFE shall be equal to the whole Angle AGH. By the same Reason, the Angle CDE is equal to the Angle ABH; and the Angle at C equal to the Angle at A; and the Angle E equal to the Angle H. Therefore the Figure AH is equiangular to the Figure CE; and they have the Sides about the equal Angles proportional. Conse-

* Def. 1. of this. quently, the Right-lined Figure AH will be * similar to the Right-lined Figure CE. Therefore, *there is described upon the given Right Line* AB *the Right-lined Figure* AH, *similar, and similarly situate, to the given Right-lined Figure* CE; which was to be done.

PROPOSITION XIX.

THEOREM.

Similar Triangles are in the duplicate Proportion of their homologous Sides.

LET ABC, DEF, be similar Triangles, having the Angle B equal to the Angle E; and let AB be to BC, as DE is to EF, so that BC be the Side homologous to EF. I say, the Triangle ABC, to the Triangle DEF, has a duplicate Proportion to that of the Side BC to the Side EF.

* 11 of this. For, take * BG a third Proportion to BC and EF; that is, let BC be to EF as EF is to BG, and join GA.

Then, because AB is to BC, as DE is to EF; it shall be (by Alternation), as AB is to DE, so is BC to EF; but as BC is to EF, so is EF to BG. There-

‡ 11. 5. fore, as AB is to DE, so is ‡ EF to BG: Consequently, the Sides that are about the equal Angles of

 the

the Triangles ABG, DEF, are reciprocal: But those Triangles that have one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles reciprocal, are ‡ equal. Therefore the Triangle ABG is equal to the Triangle DEF; and because BC is to EF, as EF is to BG; and if three Right Lines be proportional, the first has * a duplicate Proportion to the third, of what it has to the second; BC to BG shall have a duplicate Proportion of that which BC has to EF; and as BC is to BG, so is the Triangle ABC to the Triangle ABG; whence the Triangle ABC bears to the Triangle ABG a duplicate Proportion to what BC doth to EF; but the Triangle ABG is equal to the Triangle DEF. Therefore the Triangle ABC, to the Triangle DEF, shall be in the duplicate Proportion of that which the Side BC has to the Side EF. Wherefore, *similar Triangles are in the duplicate Proportion of their homologous Sides*; which was to be demonstrated.

‡ 15 *of this.*

* *Def.* 10. 5.

Coroll. From hence it is manifest, if three Right Lines be proportional; then, as the first is to the third, so is a Triangle made upon the first, to a similar and similarly described Triangle upon the second; because it has been proved, that as CB is to BG, so is the Triangle ABC to the Triangle ABG; that is, to the Triangle DEF; *which was to be demonstrated.*

PROPOSITION XX.

Theorem.

Similar Polygons are divided into similar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other.

LET ABCDE, FGHKL, be similar Polygons, and let the Side AB be homologous to the Side FG. I say, the Polygons ABCDE, FGHKL, are divided into equal Numbers of similar Triangles, and homologous to the Wholes; and the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of that which the Side AB has to the Side FG.

For let BE, EC, GL, LH, be joined.

Then,

Then, becaufe the Polygon ABCDE is fimilar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL. Now, fince ABE, FGL, are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles proportional; the
6 *of this.* Triangle ABE will be * equiangular to the Triangle FGL, and alfo fimilar to it. Therefore the Angle ABE is equal to the Angle FGL; but the whole An-
Def. 1. *of this.* gle ABC is * equal to the whole Angle FGH, becaufe of the Similarity of the Polygons; therefore the remaining Angle EBC is equal to the remaining Angle LGH: (And fince by the Similarity of the Triangles ABE, FGL), as EB is to BA, fo is LG to GF; and fince, alfo, by the Similarity of the Polygons, AB is to
22. 5. BC as FG is to GH; it fhall be, by Equality of Proportion, as EB is to BC, fo is LG to GH; that is, the Sides about the equal Angles EBC, LGH, are proportional. Wherefore the Triangle EBC is equiangular to the Triangle LGH, and, confequently, alfo fimilar to it. For the fame Reafon, the Triangle ECD is likewife fimilar to the Triangle LHK; therefore the fimilar Polygons ABCDE, FGHKL, are divided into equal Numbers of fimilar Triangles.

I fay, they are alfo homologous to the Wholes; that is, that the Triangles are proportional and the Antecedents are ABE, EBC, ECD; and their Confequents FGL, LGH, LHK. And the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of an homologous Side of the one, to an homologous Side of the other; that is, as AB to FG.

For, becaufe the Triangle ABE is fimilar to the
19 *of this.* Triangle FGL, the Triangle ABE fhall be * to the Triangle FGL, in the duplicate Proportion of BE to GL: For the fame Reafon, the Triangle BEC, to the Triangle GLH, is * in a duplicate Proportion of BE to GL: Therefore the Triangle ABE is † to
11. 5. the Triangle FGL, as the Triangle BEC is to the Triangle GLH. Again, becaufe the Triangle EBC, is fimilar to the Triangle LGH, the Triangle EBC, to the Triangle LGH, fhall be in the duplicate Proportion of the Right Line CE to the Right Line HL; and fo, likewife, the Triangle ECD to the Triangle

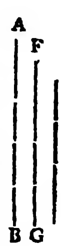
angle LHK, ſhall be in the duplicate Proportion of CE to HL. Therefore the Triangle BEC is to the Triangle LGH, as the Triangle CED is to the Triangle LHK. But it has been proved, that the Triangle EBC is to the Triangle LGH, as the Triangle ABE is to the Triangle FGL. Therefore, as the Triangle ABE is to the Triangle FGL, ſo is the Triangle BEC to the Triangle GHL; and ſo is the Triangle ECD to the Triangle LHK. But as one of the Antecedents is to one of the Conſequents, ſo are ‡ all the Antecedents to all the Conſequents. Wherefore, as the Triangle ABE is to the Triangle FGL, ſo is the Polygon ABCDE to the Polygon FGHKL: But the Triangle ABE, to the Triangle FGL, * is in the duplicate Proportion of the homologous Side AB to the homologous Side FG; for similar Triangles are in the duplicate Proportion of the homologous Sides. Wherefore the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of the homologous Side AB to the homologous Side FG. Therefore, *ſimilar Polygons are divided into ſimilar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other*; which was to be demonſtrated.

‡ 12. 5.

* 19 *of this.*

It may be demonſtrated, after the ſame manner, that ſimilar quadrilateral Figures are to each other in the duplicate Proportion of their homologous Sides; and this has been already proved in Triangles.

Coroll. 1. Therefore, univerſally, ſimilarly Right-lined Figures are to one another in the duplicate Proportion of their homologous Sides; and if X be taken a third Proportional to AB and FG, then AB will have to X a duplicate Proportion of that which AB has to FG; and a Polygon to a Polygon, and a quadrilateral Figure to a quadrilateral Figure, will be in the duplicate Proportion of that which one homologous Side has to the other; that is, AB to FG; but this has been proved in Triangles.



2. Therefore, univerſally, it is manifeſt, if three Right Lines be proportional, as the firſt is to the third, ſo is a Figure deſcribed upon the firſt, to a ſimilar and

ſimilarly

similarly described Figure on the second; *which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

Figures that are similar to the same Right-lined Figure, are also similar to one another.

LET each of the Right lined Figures A, B, be similar to the Right-lined Figure C. I say, the Right-lined Figure A is also similar to the Right lined Figure B.

For, because the Right-lined Figure A is similar to the Right-lined Figure C, it shall be * equiangular thereto; and the Sides about the equal Angles proportional. Again, because the Right-lined Figure B is similar to the Right-lined Figure C, it shall * be equiangular thereto; and the Sides about the equal Angles will be proportional. Therefore each of the Right lined Figures A, B, are equiangular to C, and they have the Sides about the equal Angles proportional. Wherefore the Right-lined Figure A is equiangular to the Right-lined Figure B; and the Sides about the equal Angles are proportional. Wherefore, A *is similar to* B; which was to be demonstrated.

* *Def. 1. of this.*

PROPOSITION XXII.

THEOREM.

If four Right Lines be proportional, the Right-lined Figures, similar and similarly described upon them, shall be proportional; and if similar Right-lined Figures similarly described upon Lines be proportional, then the Right Lines shall be also proportional.

LET four Right Lines AB, CD, EF, GH, be proportional; and as AB is to CD, so let EF be to GH.

Now, let the similar Figures KAB, LCD, be similarly described * upon AB, CD; and the similar Figures

* 18 *of this.*

Figures MF, NH, ſimilarly deſcribed upon the Right Lines EF, GH. I ſay, as the Right-lined Figure KAB is to the Right-lined Figure LCD, ſo is the Right-lined Figure MF to the Right-lined Figure NH.

For, take * X a third Proportional to AB, CD; * 11 *of this.* and O a third Proportional to EF, GH.

Then, becauſe AB is to CD, as EF is to GH; and as CD is to X, ſo is GH to O; it ſhall be †, by † 22. 5. Equality of Proportion, as AB is to X, ſo is EF to O. But AB is to X, as the Right-l[illegible]

[illegible] to the Right-lined Figure NH. Therefore, as the Right-lined Figure KAB is to the Right-lined Figure LCD, ſo is * the * 11. Right-lined Figure MF to the Right-lined Figure NH.

And, if the Right-lined Figure KAB be to the Right-lined Figure LCD, as the Right-lined Figure MF is to the Right-lined Figure NH; I ſay, as AB is to CD, ſo is EF to GH.

For, make † EF to PR, as AB is to CD, and deſcribe upon PR a Right-lined Figure SR ſimilar, and alike ſituate, to either of the Figures MF, NH. † 12 *of this.*

Then, becauſe AB is to CD, as EF is to PR; and there are deſcribed upon AB, CD, ſimilar and alike ſituate Right-lined Figures KAB, LCD; and upon EF, PR, ſimilar and alike ſituate Figures MF, SR; it ſhall be (by what has been already proved), as the Right-lined Figure KAB is to the Right-lined Figure LCD, ſo is the Right-lined Figure MF to the Right-lined Figure SR: But (by the *Hyp.*) as the Right-lined Figure KAB is to the Right-lined Figure LCD, ſo is the Right-lined Figure MF to the Right-lined Figure NH. Therefore, as the Right-lined Figure MF is to the Right-lined Figure NH, ſo is the Right-lined Figure MF to the Right-lined Figure SR: And ſince the Right-lined Figure MF has the ſame Proportion to NH, as it hath to SR, the Right-lined Figure NH ſhall be ‡ equal to the Right-lined Figure SR; it is ‡ 9. alſo ſimilar to it, and alike deſcribed; therefore GH is equal to PR. And becauſe AB is to CD, as EF is to PR; and PR is equal to GH; it ſhall be as AB is to CD, ſo is EF to GH. Therefore, *if four Right Lines be proportional, the Right-lined Figures, ſimilar and ſimilarly deſcribed upon them, ſhall be proportional;*

and

and if similar Right-lined Figures, similarly described upon Lines, be proportional, then the Right Lines shall also be proportional; which was to be demonstrated.

LEMMA.

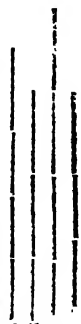
Any three Right Lines A, B, *and* C, *being given, the Ratio of the first* A, *to the third* C, *is equal to the Ratio compounded of the Ratio of the first* A *to the second* B, *and of the Ratio to the second* B *to the third* C.

A B C

FOR Example, Let the Number 3 *be the Exponent or Denominator of the Ratio of* A *to* B; *that is, let* A *be three Times* B, *and let the Number* 4 *be the Exponent of the Ratio of* B *to* C; *then the Number* 12, *produced by the Multiplication of* 4 *and* 3, *is the compounded Exponent of the Ratio of* A *to* C: *For, since* A *contains* B *thrice, and* B *contains* C *four Times,* A *will contain* C *thrice four Times, that is,* 12 *Times. This is also true of other Multiples, or Submultiples; but this Theorem may be universally demonstrated thus: The Quantity of the Ratio of* A *to* B, *is the Number* $\frac{A}{B}$; viz. *which, multiplying the Consequent, produced the Antecedent. So likewise the Quantity of the Ratio of* B *to* C, *is* $\frac{B}{C}$. *And these two Quantities, multiplied by each other, produce the Number* $\frac{A \times B}{B \times C}$, *which is the Quantity of the Ratio that the Rectangle, comprehended under the Right Lines* A *and* B, *has to the Rectangle comprehended under the Right Lines* B *and* C; *and so the said Ratio of the Rectangle under* A *and* B, *to the Rectangle under* B *and* C, *is that which, in the Sense of* Def. 5. *of this Book, is compounded of the Ratios of* A *to* B, *and* B *to* C; *but* (*by* 1. 6.) *the Rectangle contained under* A *and* B, *is to the Rectangle contained under* B *and* C, *as* A *is to* C; *therefore the Ratio of* A *to* C, *is equal to the Ratio compounded of the Ratios of* A *to* B, *and of* B *to* C.

If any four Right Lines A, B, C, *and* D, *be proposed, the Ratio of the first* A *to the fourth* D, *is equal to the Ratio compounded of the Ratio of the first* A *to the second* B, *and of the Ratio of the second* B *to the third* C, *and of the Ratio of the third* C *to the fourth* D.

For, in three Right Lines A, C, *and* D, *the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratios of* A *to* C, *and of* C *to* D; *and it has been already demonstrated, that the Ratio of* A *to* C *is equal to the Ratio compounded of the Ratios of* A *to* B, *and of* B *to* C. *Therefore the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratios of* A *to* B, *of* B *to* C, *and of* C *to* D. *After the same manner we demonstrate, in any Number of Right Lines, that the Ratio of the first to the last is equal to the Ratio compounded of the Ratios of the first to the second, of the second to the third, of the third to the fourth, and so on to the last.*


A B C D

This is true of any other Quantities besides Right Lines, which will be manifest, if the same Number of Right Lines A, B, C, &c. *as there are Magnitudes, be assumed in the same Ratio;* viz. *so that the Right Line* A *is to the Right Line* B, *as the first Magnitude is to the second, and the Right Line* B *to the Right Line* C, *as the second Magnitude is to the third, and so on. It is manifest (by 22. 5.), by Equality of Proportion, that the first Right Line* A *is to the last Right Line, as the first Magnitude is to the last; but the Ratio of the Right Line* A *to the last Right Line is equal to the Ratio compounded of the Ratios of* A *to* B, B *to* C, *and so on to the last Right Line: But (by the* Hyp.) *the Ratio of any one of the Right Lines to that nearest to it, is the same as the Ratio of a Magnitude of the same Order to that nearest it. And therefore the Ratio of the first Magnitude to the last, is equal to the Ratio compounded of the Ratios of the first Magnitude to the second, of the second to the third, and so on to the last;* which was to be demonstrated.

PROPOSITION XXIII.

THEOREM.

Equiangular Parallelograms have the Proportion to one another that is compounded of their Sides.

LET AC, CF, be equiangular Parallelograms, having the Angle BCD equal to the Angle ECG. I say, the Parallelogram AC to the Parallelogram CF, is in the Proportion compounded of their Sides; *viz.* compounded of the Proportion of BC to CG, and of DC to CE.

For, let BC be placed in the same Right Line with CG.

* 14. 1. Then DC shall be * in a strait Line with CE, and
† 12 of this. compleat the Parallelogram DG; and then †, as BC is to CG, so is some Right Line K to L; and as DC is to CE, so let L be to M.

Then the Proportions of K to L, and of L to M, are the same as the Proportions of the Sides; *viz.* of BC to CG, and DC to CE; but the Proportion of
‡ Lemma preced. K to M is ‡ compounded of the Proportion of K to L, and of the Proportion of L to M. Therefore, also, K to M hath a Proportion compounded of the Sides. Then, because BC is to CG as the Parallelogram AC
* 1 of this. is * to the Parallelogram CH: And since BC is to
† 11. 5. CG, as K is to L; it shall be †, as K is to L, so is the Parallelogram AC to the Parallelogram CH. Again, because DC is to CE, as the Parallelogram CH is to the Parallelogram CF; and since as DC is to CE, so is L to M; therefore as L is to M, so shall † the Parallelogram CH be to the Parallelogram CF: And consequently since it has been proved that K is to L, as the Parallelogram AC is to the Parallelogram CH; and as L is to M, so is the Parallelogram CH to the
‡ 22. 5. Parallelogram CF; it shall be ‡, by Equality of Proportion, as K is to M, so is the Parallelogram AC to the Parallelogram CF; but K to M hath a Proportion compounded of the Sides: Therefore, also, the Parallelogram AC, to the Parallelogram CF, hath a Proportion compounded of the Sides. Wherefore, *equiangular Parallelograms have the Proportion to one an-*

other

other that is compounded of their Sides; which was to be demonstrated.

PROPOSITION XXIV.

THEOREM.

In every Parallelogram, the Parallelograms that are about the Diameter, are similar to the Whole, and also to one another.

LET ABCD be a Parallelogram, whose Diameter is AC; and EG, HK, be Parallelograms about the Diameter AC. I say, the Parallelograms EG, HK, are similar to the Whole ABCD, and also to each other.

For, because EF is drawn parallel to BC, the Side of the Triangle ABC, it shall be *, as BE is to EA, so is CF to FA. Again, because FG is drawn parallel to CD, the Side of the Triangle ACD, it shall be as CF is to FA, so is * DG to GA. But CF is to FA (as has been proved), as BE is to EA. Therefore, as BE is to EA, so is † DG to GA; and by compounding, as BA is to AE, so is ‡ DA to AG; and, by Alternation, as BA is to AD, so is AE to AG. Therefore the Sides of the Parallelograms ABCD, EG, which are about the common Angle BAD are proportional. And because GF is parallel to DC, the Angle AGF is * equal to the Angle ADC, and the Angle GFA equal to the Angle DCA; and the Angle DAC is common to the two Triangles ADC, AGF. Wherefore the Triangle ADC will be equiangular to the Triangle AGF. For the same Reason, the Triangle ACB is equiangular to the Triangle AFE. Therefore the whole Parallelogram ABCD is equiangular to the Parallelogram EG; and so AD is to DC, as AG is † to GF; DC is to CA, as GF is to FA; and AC is to CB, as AF is to FE; and, moreover, CB is to BA, as FE is to to EA. Wherefore, since it has been proved, that DC is to CA, as GF is to FA; and AC it to CB, as AF is to FE; it shall be, by Equality of Proportion, as DC is to CB, so is GF to FE. Therefore the Sides that are about the equal Angles of the Parallelograms ABCD, EG,

* 2 of this.
† 11. 5.
‡ 18. 5.
* 29. 1.
† 4 of this.

are proportional; and, accordingly, the Parallelogram ABCD is ſimilar to the Parallelogram EG. For the ſame Reaſon, the Parallelogram ABCD is ſimilar to the Parallelogram KH. Therefore both the Parallelograms EG, HK, are ſimilar to the Parallelogram ABCD. But Right-lined Figures that are ſimilar to the † ſame Right-lined Figure, are ſimilar to one another. Therefore the Parallelogram EG is ſimilar to the Parallelogram HK. And ſo, *in every Parallelogram, the Parallelograms that are about the Diameter, are ſimilar to the Whole, and alſo to one another*; which was to be demonſtrated.

† 21 *of this.*

PROPOSITION XXV.

PROBLEM.

To deſcribe a Right-lined Figure ſimilar to a Right-lined Figure which ſhall be given, and equal to another Right-lined Figure given.

LET ABC, and D, be two given Right-lined Figures; it is required to deſcribe another Figure, ſimilar to ABC and equal to D.

* 44. 1. On the Side BC of the given Figure ABC *, make the Parallelogram BE equal to the Right-lined Figure ABC; and on the Side CE make * the Parallelogram CM equal to the Right-lined Figure D, in the Angle FCE, equal to the Angle CBL. Then BC, CF, as alſo LE, EM, will be † in two ſtrait Lines. Find ‡ GH a mean Proportional between BC and CF; and on GH let there be deſcribed * the Right-lined Figure KGH, ſimilar, and alike ſituate, to the Right-lined Figure ABC.

† 14. 1.
‡ 13 *of this.*
* 18 *of this.*

And then, becauſe BC is to GH, as GH is to CF; and ſince, when three Right Lines are proportional, the firſt is to the third, as the Figure deſcribed on the firſt is † to a ſimilar and alike ſituate Figure deſcribed on the ſecond; it ſhall be, as BC is to CF, ſo is the Right-lined Figure ABC to the Right-lined Figure KGH. But as BC is to CF, ſo is ‡ the Parallelogram BE to the Parallelogram EF. Therefore, as the Right-lined Figure ABC is to the Right-lined Figure KGH, ſo is the Parallelogram BE to the Parallelogram EF.

† Cor. 20. *of this.*
‡ 1 *of this.*

Wherefore (by Alternation), as the Right-lined Figure ABC is to the Parallogram BE, so is the Right-lined Figure KGH to the Parallelogram EF. But the Right-lined Figure ABC is equal to the Parallelogram BE. Therefore the Right-lined Figure KGH is also equal to the Parallelogram EF. But the Parallelogram EF is equal to the Right-lined Figure D. Therefore the Right-lined Figure KGH is equal to D. But KGH is similar to ABC. Consequently, *there is described the Right-lined Figure* KGH *similar to the given Figure* ABC, *and equal to the given Figure* D; which was to be done.

PROPOSITION XXVI.

THEOREM.

If from a Parallelogram be taken away another similar to the Whole, and in like manner situate, having also an Angle common with it; then is that Parallelogram about the same Diameter with the Whole.

LET the Parallelogram AF be taken away from the Parallelogram ABCD, similar to ABCD, and in like manner situate, having the Angle DAB common. I say, the Parallelogram ABCD is about the same Diameter with the Parallelogram AF.

For, if it be not, let AHC be the Diameter of the Parallelogram BD, and let GF be produced to H; also let HK be drawn parallel to AD, or BC.

Then, because the Parallelogram ABCD is about the same Diameter as the Parallelogram KG, the Parallelogram ABCD shall be * similar to the Parallelogram KG; and so, as DA is to AB, so is † GA to AK. But because of the Similarity of the Parallelograms * ABCD, EG; as DA is to AB, so is GA to AE. And therefore, as GA is ‡ to AE, so is GA to AK. And since GA has the same Proportion to AK as to AE, AE is † equal to AK, the less to a greater, or the greater to a less; which is absurd. Therefore the Parallelogram ABCD is not about the same Diameter as the Parallelogram AH. And therefore it will be about the same Diameter with the Parallelogram AF.

* 24 of this
† Def. 1. of this.
* Hyp.
‡ 11. 5.
† 9. 5.

AF. Therefore, *if from a Parallelogram be taken away another ſimilar to the Whole, and in like manner ſituate, having alſo an Angle common with it; then is that Parallelogram about the ſame Diameter with the Whole;* which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

Of all Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar, and alike ſituate, to that deſcribed on the half Line, the greateſt is that which is applied to the half Line, and it is ſimilar to the Defect.

LET AB be a Right Line, biſected in the Point C; and let the Parallelogram AD be applied to the Right Line AB, wanting in Figure the Parallelogram CE, ſimilar and alike ſituate to that deſcribed on half of the Right Line AB. I ſay, AD is the greateſt of all Parallelograms applied to the Right Line AB, wanting in Figure by Parallelograms ſimilar and alike ſituate to CE. For, let the Parallelogram AF be applied to the Right Line AB, wanting in Figure the Parallelogram HK, ſimilar and alike ſituate to the Parallelogram CE. I ſay, the Parallelogram AD is greater than the Parallelogram AE.

* 26 *of this.* For, becauſe the Parallelogram CE is ſimilar to the Parallelogram HK, they ſtand * about the ſame Diameter. Let DB, their Diameter, be drawn, and the Figure deſcribed; then, ſince the Parallelogram CF is † equal to FE, let HK, which is common, be added; and the Whole CH is equal to the Whole KE. But CH is ‡ equal to CG, becauſe the Right Line AC is equal to CB; therefore CG is equal to KE; add the common Parallelogram CF, and the Whole AF is equal to the Gnomon LNM; and ſo CE, that is the Parallelogram AD, is greater than the Parallelogram AF. Therefore, *of all the Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar, and alike ſituate, to that deſcribed on the half Line, the greateſt is that which is applied to the half Line, and it is ſimilar to the Defect;* which was to be demonſtrated.

† 43. 1.

‡ 36. 1.

PRO-

PROPOSITION XXVIII.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right-lined Figure given, deficient by a Parallelogram, which is similar to another given Parallelogram; but it is necessary that the Right-lined Figure given, to which the Parallelogram to be applied must be equal, be not greater than the Parallelogram which is applied to the half Line, since the Defects must be similar, viz. *the Defect of the Parallelogram applied to the half Line, and the Defect of the Parallelogram to be applied.*

LET AB be a given Right Line, and let the given Right-lined Figure, to which the Parallelogram to be applied to the Right Line AB must be equal, be C, which must not be greater than the Parallelogram applied to the half Line, the Defects being similar; and let D be the Parallelogram, to which the Defect of the Parallelogram to be applied is similar. Now it is required to apply a Parallelogram equal to the given Right-lined Figure C to the given Right Line AB, deficient by a Parallelogram similar to D.

Let AB be bisected in E, and on EB describe * the Parallelogram EBFG, similar and alike situate to D, and compleat the Parallelogram AG. * 18 of this.

Now, AG is either equal to C, or greater than it, because of the Determination. If AG be equal to C, what was proposed will be done; for the Parallelogram AG is applied to the Right Line AB equal to the given Right-lined Figure C, deficient to the Parallelogram EF, similar to the Parallelogram D. But, if it be not equal, then HE is greater than C; but EF is equal to HE; therefore EF shall also be greater than C. Now make † the Parallelogram KLMN similar and alike situate to D, and equal to the Excess, by which EF exceeds C. But D is similar to EF; wherefore KM shall also be similar to EF. Therefore let the Right Line KL be homologous to GE, and LM to GF: Then, because EF is equal to C and KM together, EF † 25 of this.

will be greater than KM; and so the Right Line GE is greater than KL, and GF than LM. Make GX equal to KL, and GO equal to LM, and compleat the Parallelogram XGOP. Therefore XO is equal and similar to KM; but KM is similar to EF; therefore XO is * similar to EF; and so XO is † about the same Diameter with FE: Let GPB be their Diameter, and the Figure be described.

* 21 *of this.*
† 26 *of this.*

Then, since EF is equal to C and KM together, and XO is equal to KM, the Gnomon ϒΦΨ remaining is equal to the remaining Figure C; and because OR is equal to XS, let SR, which is common, be added; then the Whole OB is equal to the Whole XB; but XB is equal to TE, since the Side AT is equal to the Side EB. Wherefore TE is equal to [illegible]. Add XS, which is common, and then the Whole TS is equal to the Whole Gnomon ϒΦΨ; but the Gnomon ϒΦΨ has been proved equal to C; and TS shall be equal to C; and so, *the Parallelogram* TS *is applied to the Right Line* AB, *equal to the given Right-lined Figure* C, *and deficient by a Parallelogram* SR, *similar to the Parallelogram* D, *because* SR *is similar to* FE; which was to be done.

PROPOSITION XXIX.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right-lined Figure given, exceeding by a Parallelogram, which shall be similar to another given Parallelogram.

LET AB be a given Right Line, and let C be the given Right-lined Figure, to which that to be applied to AB must be equal: Likewise, let D be the Parallelogram, to which the exceeding Parallelogram is to be similar; it is required to apply a Parallelogram to the Right Line AB, equal to the given Right-lined Figure C, exceeding by a Parallelogram similar to D.

Bisect AB in E, and let the Parallelogram EL be described * upon the Right Line EB, similar and alike situate to D; and let † the Parallelogram GH be equal to EL and C together, but similar to D, and alike situate;

* 18 *of this.*
† 25 *of this.*

ſituate; therefore GH is ſimilar to EL. Let KH be a Side homologous to FL, and KG to FE; then becauſe the Parallelogram GH is greater than the Parallelogram EL, the Right Line KH will be greater than FL, and KG greater than FE. Let FL, FE, be produced, and let FLM be equal to KH, FEN equal to KG, and compleat the Parallelogram MN; therefore MN is equal and ſimilar to GH; but GH is ſimilar to EL, and ſo MN ſhall be ‡ ſimilar to EL; and, accordingly, EL is * about the ſame Diameter with MN. Let FX be their Diameter, and deſcribe the Figure. ‡ 21 *of this.* * 26 *of this.*

Then, ſince GH is equal to EL and C together, as likewiſe to MN; therefore MN ſhall be equal to EL and C together. Let EL, which is common, be taken away; then the Gnomon ΦΥΨ remaining is equal to C; and ſince AE is equal to EB, the Parallelogram AN will be alſo equal to the Parallelogram EP, that is, to LO; and if BX, which is common, be added, then the whole Parallelogram AX is equal to the Gnomon ΦΥΨ; but the Gnomon ΦΥΨ is equal to C; therefore AX ſhall alſo be equal to C. Wherefore *the Parallelogram* AX *is applied to the given Right Line* AB, *equal to the given Right-lined Figure* C, *and exceeding by the Parallelogram* PO, *ſimilar to the Parallelogram* D; which was to be done.

PROPOSITION XXX.

PROBLEM.

To cut a given terminate Right Line according to extreme and mean Ratio.

LET AB be a given terminate Line; it is required to cut the ſame according to extreme and mean Ratio.

Deſcribe * CB, the Square of AB; and apply the Parallelogram CD to AC, equal to the Square BC, exceeding † by the Figure AD ſimilar to BC; but BC is a Square; therefore AD ſhall alſo be a Square. * 46. 1. † 29 *of this.*

Now, becauſe BC is equal to CD, take away CE, which is common; then BF remaining ſhall be equal

to AD remaining; but BF is equiangular to AD; therefore the Sides that are about the equal Angles are
‡ 14 of this. ‡ reciprocally proportional; and so as FE is to ED,
* 34. 1. so is AE to EB; but FE is * equal to AC, that is, to AB; and ED to AE: Wherefore, as AB is to AE, so is AE to EB; but AB is greater than AE; therefore
† 14. 5. fore AE is † greater than EB; and so the Right Line AB is cut, according to extreme and mean Ratio, in the Point E; and AE is the greater Segment thereof; *which was to be done.*

Otherwise thus: Let AB be the Right Line given; it is required to cut the same into extreme and mean Ratio.

‡ 11. 2. Divide ‡ AB so in C, that the Rectangle contained under AB, BC, be equal to the Square of AC.

Then, because the Rectangle under AB, BC, is
* 17 of this. equal to the Square of AC, it shall be *, as AB is to AC, so is AC to CB; and so, *the Right Line* AB *is cut into mean and extreme Ratio*; which was to be done.

PROPOSITION XXXI.

THEOREM.

Any Figure described upon the Side of a Right-angled Triangle, subtending the Right Angle, is equal to the two Figures described upon the Sides containing the Right Angle, being similar and alike situate to the former Figure.

LET ABC be a rectangular Triangle, having the Right Angle BAC. I say, the Figure described on BC is equal to the two Figures described on BA, AC, together, which are similar and alike situate to the Figure described on BC.

For, draw the Perpendicular AD.

Then, because the Right Line AD is drawn in the Right-angled Triangle ACB, from the Right Angle A, perpendicular to the Base BC; the Triangles ABD, ADC, which are about the Perpendicular AD, will
* 8 of this. be * similar to the whole Triangle ABC, and also to each other. Then, because the Triangle ABC is similar to the Triangle ABD, it shall be *, as CB is to BA,

ſo is BA to BD: And ſince, when three Right Lines are proportional, the firſt ſhall be † to the third, as a Figure deſcribed on the firſt, to a ſimilar and alike ſituate Figure deſcribed on the ſecond; therefore, as CB is to BD, ſo is a Figure deſcribed on CB, to a ſimilar and a like ſituate Figure deſcribed on BA. For the ſame Reaſon, as BC is to CD, ſo is a Figure deſcribed on BC to one deſcribed on CA. Wherefore, alſo, as BC is to BD and DC, together, ſo is ‡ the Figure deſcribed on BC to thoſe two together, that are deſcribed ſimilar and alike ſituate on BA and AC; but BC is equal to BD and DC together: Therefore the Figure deſcribed on BC is equal to thoſe together, which are deſcribed on BA and AC, ſimilar and alike ſituate to that on BC. Wherefore, *any Figure deſcribed upon the Side of a Right-angled Triangle, ſubtending the Right Angle, is equal to the Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure*; which was to be demonſtrated.

† Cor. 20. of this.

‡ 24. 5.

PROPOSITION XXXII.

THEOREM.

If two Triangles, having two Sides proportional to two Sides, be ſo compounded, or ſet together, at one Angle, that their homologous Sides be parallel; then the other Sides of theſe Triangles will be in one ſtrait Line.

LET there be two Triangles ABC, DCE, having two Sides BA, AC, of the one, proportional to two Sides CD, DE, of the other; *viz.* let BA be to AC, as CD is to DE; alſo let AB be parallel to DC, and AC to DE. I ſay, BC, CE, are both in one ſtrait Line.

For, becauſe AB is parallel to DC, and the Right Line AC falls on them, the alternate Angles BAC, ACD, will be * equal to each other. And by the ſame Reaſon, the Angle CDE is equal to the Angle ACD; wherefore the Angle BAC is equal to the Angle CDE. Then, becauſe ABC, DCE, are two Triangles, having one Angle A equal to one Angle

* 29. 1.

D,

D, and the Sides about the equal Angles proportional, *viz.* BA to AC, as CD to DE; the Triangle ABC will be * equiangular to the Triangle DCE; wherefore the Angle ABC is equal to the Angle DCE; but the Angle ACD has been proved to be equal to the Angle BAC; therefore the whole Angle ACE is equal to the two Angles ABC, BCA; and if ACB, which is common, be added, then the Angles ACE, ACB, are equal to the Angles BAC, ACB, CBA; but the Angles BAC, ACB, CBA ‡, are equal to two Right Angles. Therefore the Angles ACE, ACB, will also be equal to two Right Angles; and so at the Point C, in the Right Line AC, two Right Lines BC, CE, tending contrary Ways, make the adjacent Angles ACE, ACB, equal to two Right Angles; therefore BC shall be † in the same Right Line with CE. Wherefore, *if two Triangles having two Sides proportional to two Sides, be so compounded or set together, at one Angle, that their homologous Sides be parallel; then the other Sides of these Triangles will be in one strait Line*; which was to be demonstrated.

* 6 of this.

‡ 32. 1.

† 14. 1.

PROPOSITION XXXIII.

THEOREM.

In equal Circles the Angles have the same Proportion with the Circumferences on which they stand, whether the Angles be at the Centres, or at the Circumferences; and so likewise are the Sectors, as being at the Centres.

LET ABC, DEF, be equal Circles; and let the Angles BGC, EHF, be at their Centres G, H; and the Angles BAC, EDF, at their Circumferences. I say, as the Circumference BC is to the Circumference EF, so is the Angle BGC to the Angle EHF; and so is the Angle BAC to the Angle EDF; and so is the Sector BGC to the Sector EHF.

For, assume any Number of continuous Circumferences CK, KL, each equal to BC; and also any Number FM, MN, each equal to EF; and join GK, GL, HM, HN.

Then,

Then, becauſe the Circumferences BC, CK, KL, are equal to each other; the Angles BGC, CGK, KGL, will be alſo * alſo equal to one another; and ſo the Circumference BL is the ſame Multiple of the Circumference BC, as the Angle BGL is of the Angle BGC. For the ſame Reaſon, the Circumference NE is the ſame Multiple of the Circumference EF, as the Angle EHN is of the Angle EHF; but if the Circumference BL be equal to the Circumference EN, then the Angle BGL ſhall be equal to the Angle EHN; and if the Circumference BL be greater than the Circumference EN, the Angle BGL will be greater than the Angle EHN; and if leſs, leſs. Therefore here are four Magnitudes, *viz.* the two Circumferences BC, EF, and the two Angles BGC, EHF; and ſince there are taken Equimultiples of the Circumference BC, and the Angle BGC, to wit, the Circumference BL, and the Angle BGL; as alſo Equimultiples of the Circumference EF, and the Angle EHF, *viz.* the Circumference EM, and the Angle EHN; and becauſe it is proved, if the Circumference BL exceeds the Circumference EN, the Angle BGL will likewiſe exceed the Angle EHN; and, if equal, equal; if leſs, leſs; it ſhall be, as the Circumference BC is to the Circumference EF, ſo is † the Angle BGC to the Angle EHF; but as the Angle BGC is to the Angle EHF, ſo is ‡ the Angle BAC to the Angle EDF, for the former are * double to the latter: Therefore, as the Circumference BC is to the Circumference EF, ſo is the Angle BGC to the Angle EHF; and ſo the Angle BAC to the Angle EDF.

* 27. 3.

† Def. 5. 5.

‡ 15. 5.

* 20. 3.

Wherefore, *in equal Circles, the Angles have the ſame Proportion as the Circumferences they ſtand on, whether they be at the Centres, or at the Circumferences.*

I ſay, moreover, that as the Circumference BC is to the Circumference EF, ſo is the Sector GBC to the Sector HFE.

For, join BC, CK, and aſſume the Points X, O, in the Circumferences BC, CK; and join BX, XC, CO, OK.

Then, becauſe the two Sides BG, GC, are equal to the two Sides CG, GK, and they contain equal Angles, the Baſe BC ſhall be † equal to the Baſe CK;

† 4. 1.

CK; as likewise the Triangle GBC to the Triangle GCK. And, because the Circumference BC is equal to the Circumference CK, and the Circumference remaining, which makes up the whole Circle ABC, is equal to the remaining Circumference, which makes
† 27. 3. up the same Circle ABC, the Angle BXC is equal † to the Angle COK; and so the Segment BXC is similar to the Segment COK; and they are upon equal Right Lines BC, CK; but similar Segments of Circles, that
* 24. 3. stand upon equal Right Lines, are * equal to each other: Therefore the Segment BXC is equal to the Segment COK. But the Triangle BGC is also equal to the Triangle CGK, and so the whole Sector BGC will be equal to the whole Sector CGK. By the same Reason, the Sector GKL will be equal to the Sector GBC, or GCK; therefore the three Sectors BGC, CGK, KGL, are equal to one another; so likewise are the Sectors HEF, HFM, HMN. Wherefore the Circumference I B is the same Multiple of the Circumference BC, as the Sector GBL is of the Sector GBC. For the same Reason, the Circumference NE is the same Multiple of the Circumference EF, as the Sector HEN is of the Sector HEF; but if the Circumference BL be equal to the Circumference EN, then the Sector BGL will be equal to the Sector EHN; and if the Circumference BL exceeds the Circumference EN, then the Sector BGL will also exceed the Sector EHN; and, if less, less. Therefore, since there are four Magnitudes, to wit, the two Circumferences BC, EF, and the two Sectors GBC, EHF; and there are taken the Circumference BL, and the Sector GBL, Equimultiples of the Circumference CB, and the Sector GBC; as also the Circumference EN, and the Sector HEN Equimultiples of the Circumference EF, and the Sector HEF; and because it is proved, that, if the Circumference BL exceeds the Circumference EN, the Sector BGL will also exceed the Sector EHN; and, if equal,
† Def. 5. 5. equal; if less, less; therefore, † *as the Circumference* BC *is to the Circumference* EF, *so is the Sector* GBC *to the Sector* HEF; which was to be demonstrated.

Coroll. 1. An Angle at the Centre of a Circle is to four Right Angles, as an Arc on which it stands is to

to the whole Circumference; for as the Angle BAC is to a Right Angle, ſo is the Arc BC to a Quadrant of the Circle: Wherefore, if the Conſequents be quadrupled, the Angle BAC ſhall be to four Right Angles as the Arc BC is to the whole Circumference.

2. The Arcs IL, BC, of unequal Circles, which ſubtend equal Angles, whether at their Centres, or Circumferences, are ſimilar; for IL, is to the whole Circumference ILE, as the Angle IAL is to four Right Angles; but as IAL, or BAC, is to four Right Angles, ſo is the Arc BC to the whole Circumference BCF. Therefore, as IL is to the whole Circumference ILE, ſo is BC to the whole Circumference BCF; and ſo the Arcs IL, BC, are ſimilar.

3. Two Semidiameters AB, AC, cut off ſimilar Arcs IL, BC, from concentric Circumferences.

The END *of the* SIXTH BOOK.

EUCLID's ELEMENTS.

BOOK XI.

DEFINITIONS.

I. *A Solid is that which has Length, Breadth, and Thickneſs.*

II. *The Term of a Solid is a Superficies.*

III. *A Right Line is perpendicular to a Plane, when it makes Right Angles with all the Lines that touch it, and are drawn in the ſaid Plane.*

IV. *A Plane is perpendicular to a Plane, when all the Right Lines in one Plane, drawn at Right Angles to the common Section of the two Planes, are at Right Angles to the other Plane.*

V. *The Inclination of a Right Line to a Plane is the acute Angle contained under that Line, and another Right one drawn in the Plane from that End of the inclining Line which is in the Plane, to the Point where a Right Line falls from the other End of the inclining Line perpendicular to the Plane.*

VI. *The Inclination of a Plane to a Plane, is the acute Angle contained under the Right Lines drawn in both the Planes to the ſame Point of their*

their common Interſection, and making Right Angles with it.

VII. *Planes are ſaid to be inclined ſimilarly, when the ſaid Angles of Inclination are equal*

VIII. *Parallel Planes are ſuch, which, being produced, never meet.*

IX. *Similar ſolid Figures are ſuch, as are contained under equal Numbers of ſimilar Planes.*

X. *Equal and ſimilar ſolid Figures are thoſe, that are contained under equal Numbers of ſimilar and equal Planes.*

XI. *A ſolid Angle is the Inclination of more than two Right Lines, that touch one another, and are not in the ſame Superficies: Or, a ſolid Angle is that which is contained under more than two plane Angles, which are not in the ſame Superficies, but being all at one Point.*

XII. *A Pyramid is a ſolid Figure comprehended under divers Planes ſet upon one Plane, and put together at one Point.*

XIII. *A Priſm is a ſolid Figure contained under Planes, whereof the two oppoſite are equal, ſimilar, and parallel, and the other Parallelograms.*

XIV. *A Sphere is a ſolid Figure, made when the Diameter of a Semicircle remaining at Reſt, the Semicircle is turned about till it returns to the ſame Place from whence it begun to move.*

XV. *The Axis of a Sphere is that fixed Line, about which the Semicircle is turned.*

XVI. *The Centre of a Sphere is the ſame with that of the Semicircle.*

XVII. *The Diameter of a Sphere is a Right Line drawn through the Centre, and terminated on either Side by the Superficies of the Sphere.*

XVIII. *A Cone is a Figure deſcribed when one of the Sides of a Right-angled Triangle, containing the Right Angle, remaining fixed, the Triangle is turned about until it returns to the Place from*

whence

whence it first began to move. And if the fixed Right Lines be equal to each other, that contains the Right Angle, then the Cone is a rectangular Cone; but, if it be less, it is an obtuse-angled Cone; if greater, an acute-angled Cone.

XIX. *The Axis of a Cone is that fixed Right Line, about which the Triangle is moved.*

XX. *The Base of a Cone is the Circle described by the Right Line moved about.*

XXI. *A Cylinder is a Figure described by the Motion of a Right-angled Parallelogram, one of the Sides containing the Right Angle, remaining fixed while the Parallelogram is turned about to the same Place from whence it began to be moved.*

XXII. *The Axis of a Cylinder is that fixed Right Line, about which the Parallelogram is turned.*

XXIII. *And the Bases of a Cylinder are the Circles that are described by the Motion of the two opposite Sides of the Parallelogram.*

XXIV. *Similar Cones and Cylinders are such, whose Axes and Diameters of their Bases are proportional.*

XXV. *A Cube is a solid Figure contained under six equal Squares.*

XXVI. *A Tetrahedron is a solid Figure contained under four equal equilateral Triangles.*

XXVII. *An Octahedron is a solid Figure contained under eight equal equilateral Triangles.*

XXVIII. *A Dodecahedron is a solid Figure contained under twelve equal equilateral and equiangular Pentagons.*

XXIX. *An Icosahedron is a solid Figure contained under twenty equal equilateral Triangles.*

XXX. *A Parallelopipedon is a Figure contained under six quadrilateral Figures, whereof those which are opposite are parallel.*

PRO-

PROPOSITION I.

Theorem.

One Part of a Right Line cannot be in a plane Superficies, and another Part above it.

FOR, if possible, let the Part AB of the Right Line ABC be in a plane Superficies, and the Part BC above the same.

There will be some Right Line in the aforesaid Plane, which, with AB, will be but one strait Line. Let this Line be DB.

Then the two given Right Lines ABC, ABD, have one common Segment AB, which is impossible; for one Right Line will not meet another in more Points than one. Wherefore, *one Part of a Right Line cannot be in a plane Superficies, and another Part above it*; which was to be demonstrated.

PROPOSITION II.

Theorem.

If two Right Lines cut each other, they are both in one Plane; and every Triangle is in one Plane.

LET two Right Lines AB, CD, cut each other in the Point E. I say, they are both in one Plane; and every Triangle is one Plane.

For, take any Points F and G, in the Right Lines AB, CD; and join CB, FG; and let there be drawn FH, GK. In the first Place, I say, the Triangle EBC is in one Plane.

For, if one Part FHC, or GBK, of the Triangle EBC, be in one Plane, and the other Part in another Plane; then one Part of each of the Lines EC, EB, shall be in one Plane, and the other Part in another Plane; which we have proved * to be absurd. Therefore the Triangle EBC is in one Plane; but both the Right Lines EC, EB, are in the same Plane as the * 1 of this.

Triangle BCE is; and AB, CD, are both in the same Plane as EC, EB, are. Wherefore, *the Right Lines* AB, CD, *are both in one Plane; and every Triangle is in one Plane*; which was to be demonstrated.

PROPOSITION III.

THEOREM.

If two Planes cut each other, their common Section will be a Right Line.

LET two Planes AB, BC, cut each other, whose common Section is the Line DB; I say, DB is a Right Line.

For if it be not, draw the Right Line DEB in the Plane AB, from the Point D to the Point B, and the Right Line DFB in the Plane BC.

Then two Right Lines DEB, DFB, have the same Terms, and include a Space, which is * absurd. Therefore DEB, DFB, are not Right Lines. In the same manner we demonstrate, that no other Line drawn from the Point D to the Point B, is a Right Line, besides DB, the common Section of the Planes AB, BC. *If*, therefore, *two Planes cut each other, their common Section will be a Right Line*; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If to two Right Lines, cutting one another, a third stands at Right Angles in the common Section, it shall be also at Right Angles to the Plane drawn thro' the said Lines.

LET the Right Line EF stand at Right Angles to the two Right Lines AB, CD, in the common Section E. I say, EF is also at Right Angles to the Plane drawn through AB, CD.

For,

For, take the equal Right Lines EA, EB, CE, DE; and thro' E any how draw the Right Line GEH, and join AD, CB; and from the Point F, let there be drawn FA, FG, FD, FC, FH, FB: Then, because two Right Lines AE, ED, are equal to two Right Lines CE, EB, and they contain * the equal Angles AED, CEB; the Base AD shall be † equal to the Base CB, and the Triangle AED equal to the Triangle CEB; and so, likewise, is the Angle DAE equal to the Angle EBC; but the Angle AEG is * equal to the Angle BEH; therefore AGE, BEH, are two Triangles, having two Angles of the one equal to two Angles of the other, each to each, and one Side AE equal to one Side EB; *viz.* those that are at the equal Angles; and so the other Sides of the one will be ‡ equal to the other Sides of the other. Therefore GE is equal to GH, and AG to BH; and since AE is equal to EB, and FE is common and at Right Angles, the Base AF shall be † equal to the Base FB: For the same Reason, likewise, shall CF be equal to FD. Again, because AD is equal to CB, and AF to FB, the two Sides FA, AD, will be equal to the two Sides FB, BC, each to each; but the Base DF has been proved equal to the Base FC. Therefore the Angle FAD is * equal to the Angle FBC: Moreover, AG has been proved equal to BH; but FB also, is equal to AF, therefore the two Sides FA, AG, are equal to the two Sides FB, BH; and the Angle FAG is equal to the Angle FBH, as has been demonstrated; wherefore the Base GF † is equal to the Base FH. Again, because GE has been proved equal to EH, and EF is common, the two Sides GE, EF, are equal to the two Sides HE, EF; but the Base HF is equal to the Base FG; therefore the Angle GEF is * equal to the Angle HEF; and so both the Angles GEF, HEF, are Right Angles: Therefore FE makes Right Angles with GH, which is any how drawn thro' E. After the same manner we demonstrate, that FE is at Right Angles to all Right Lines that are drawn in the Plane to it; but a Right Line is * at Right Angles to a Plane, when it is at Right Angles to all Right Lines drawn to it in the Plane. Therefore FE is at Right Angles to a Plane drawn thr[illegible] the Right Lines

* 15. 1.

† 4. 1.

‡ 26. 1.

† 4. 1.

* 8. 1.

* Def. 3. of this.

AB, CD. Wherefore, *if two Right Lines, cutting one another, a third ſtands at Right Angles in the common Section, it ſhall be alſo at Right Angles to the Plane drawn thro' the ſaid Lines*; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

If to three Right Lines, touching one another, a fourth ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane.

LET the Right Line AB ſtand at Right Angles, in the Point of Contact B, to the three Right Lines BC, BD, BE. I ſay, BC, BD, BE, are in one and the ſame Plane.

For, if they are not, let BD, BE, be in one Plane, and BC above it; and let the Plane paſſing thro' AB, BC, be produced, and it will * make the common Section, with the other Plane, a ſtrait Line, which let be BF; then three Right Lines AB, BC, BF, are in one Plane drawn thro' AB, BC: And ſince AB ſtands at Right Angles to BD and BE, it ſhall be † at Right Angles to a Plane drawn thro' BE, DB; and ſo AB ſhall make ‡ Right Angles with all Right Lines touching it that are in the ſame Plane: But BF, being in the ſaid Plane, touches it; wherefore the Angle ABF is a Right Angle: But the Angle ABC (by the *Hyp.*) is alſo a Right Angle; therefore the Angle ABF is equal to the Angle ABC, and they are both in the ſame Plane, which cannot be; and ſo the Right Line BC, is not above the Plane paſſing thro' BE and BD. Wherefore the three Right Lines BC, BD, BE, are is one and the ſame Plane. Therefore, *if to three Right Lines, touching one another, a fourth ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane*; which was to be demonſtrated.

* 3 *of this.*

† 4 *of this.*

‡ *Def.* 3.

PRO-

PROPOSITION VI.

THEOREM.

two Right Lines be perpendicular to one and the same Plane, those Right Lines are parallel to one another.

LET two Right Lines AB, CD, be perpendicular to one and the same Plane. I say, AB is parallel to CD.

For, let them meet the Plane in the Points B, D; and join the Right Line BD, to which let DE be drawn in the same Plane, at Right Angles, make DE equal to AB; and join BE, AE, AD.

Then because AB is at Right Angles to the aforesaid Plane, it shall be * at Right Angles to all Right Lines, touching it, drawn in the Plane; but AB touches BD, BE, which are in the said Plane; therefore each of the Angles ABD, ABE, is a Right Angle. So, for the same Reason, likewise, is each of the Angles CDB, CDE, a Right Angle. Then, because AB is equal to DE, and BD is common; the two Sides AB, BD, shall be equal to the two Sides ED, DB; but they contain Right Angles: Therefore the Base AD is † equal to the Base BE. Again, because AB is equal to DE, and AD to BE; the two Sides AB, BE, are equal to the two Sides ED, DA; but AE, their Base, is common; wherefore the Angle ABE is ‡ equal to the Angle EDA. But ABE is a Right Angle; therefore EDA is also a Right Angle; and so ED is perpendicular to DA: But it is also perpendicular to BD and DC; therefore ED is at Right Angles, in the Point of Contact, to three Right Lines BD, DA, DC: Wherefore these three last Right Lines are * in one Plane. But BD, DA, are in the same Plane as AB is; for every Triangle is † in the same Plane; therefore it is necessary, that AB, BD, DC, be in one Plane. But both the Angles ABD, BDC, are Right Angles; wherefore AB is ‡ parallel to CD. Therefore, *if two Right Lines be perpendicular to one and the same Plane, those Right Lines are parallel to one another*; which was to be demonstrated.

* Def. 3. of this.

† 4. 1.

‡ 8. 1.

* 5 of this.

† 2 of this.

‡ 28. 1.

PROPOSITION VII.

THEOREM.

If there be two parallel Lines, and any Points be taken in both of them, the Right Line joining those Points shall be in the same Plane as the Parallels are.

LET AB, CD, be two parallel Right Lines, in which are taken any Points E, F. I say, a Right Line joining the Point E, F, is in the same Plane as the Parallels are.

For, if it be not, let it be elevated above the same, if possible, as EGF, thro' which let some Plane be drawn, whose Section, with the Plane in which the

* 3 of this. Parallels are, let * be the Right Line EF; then the two Right Lines EGF, EF, will include a Space,

† Ax. 10. 1. which is † absurd: Therefore a Right Line, drawn from the Point E to the Point F, is not elevated above the Plane; and, consequently, it must be in that passing thro' the Parallels AB, CD. Wherefore, *if there be two parallel Lines, and any Points be taken in both of them, the Right Line joining those Points shall be in the same Plane as the Parallels are*; which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If there be two parallel Right Lines, one of which is perpendicular to some Plane; then shall the other be perpendicular to the same Plane.

See the Fig. of Prop. VI. LET AB, CD, be two parallel Right Lines, one of which, as AB, is perpendicular to some Plane. I say, the other, CD, is also perpendicular to the same Plane.

For, let AB, CD, meet the Plane in the Points B,

* 7 of this. D; and let BD be joined; then AB, CD, BD, are * in one Plane. Let DE be drawn in the other Plane, at

Right

Right Angles to BD, and make DB equal to AB; and join BE, AE, AD: Then, since AB is perpendicular to the Plane, it will * be perpendicular to all Right Lines touching it, that are drawn in the same Plane; therefore each of the Angles ABD, ABE, is a Right Angle. And since the Right Line BD falls on the Right Lines AB, CD; the Angles ABD, CDB, shall be † equal to two Right Angles: Therefore the Angle CDB is also a Right Angle; and so CD is perpendicular to DB. And since AB is equal to DE, and BD is common; the two Sides AB, BD, are equal to the two Sides ED, DB. But the Angle ABD is equal to the Angle EDB; for each of them is a Right Angle; therefore the Base AD is ‡ equal to the Base BE. Again, since AB is equal to DE, and BE to AD; the two Sides AB, BE, shall be equal to the two Sides ED, DA, each to each: But the Base AE is common; wherefore the Angle ABE is * equal to the Angle EDA: But the Angle ABE is a Right Angle; therefore EDA is also a Right Angle, and ED is perpendicular to DA: But it is likewise perpendicular to DB; therefore ED shall also be † perpendicular to the Plane passing thro' BD, DA, and, likewise, shall be ‡ at Right Angles to all Right Lines, drawn in the said Plane that touch it. But DC is in the Plane passing thro' BD, DA, because AB, BD, are * in that Plane; and DC is † in the same Plane that AB and BD are in; wherefore ED is at Right Angles to DC, and so CD is at Right Angles to DE, as also to DB. Therefore, CD *stands at Right Angles, in the common Section* D, *to two Right Lines* DE, DB, *mutually cutting one another; and, accordingly, is at Right Angles to the Plane passing thro'* DE, DB, which was to be demonstrated.

* Def. 3.

† 29. 1.

‡ 4. 1.

* 8. 1.

† 4 of this.

‡ Def. 3.

* 2 of this.

† 7 of this.

PRO-

PROPOSITION IX.

THEOREM.

Right Lines that are parallel to the same Right Line, not being in the same Plane with it, are also parallel to each other.

LET both the Right Lines AB, CD, be parallel to the Right Line EF, not being in the same Plane with it. I say, AB is parallel to CD.

For assume any Point G in EF, from which Point G let GH be drawn, at Right Angles to EF, in the Plane passing thro' EF, AB: Also let GK be drawn at Right Angles to EF in the Plane passing thro' EF, CD: Then, because EF is perpendicular to GH and GK, the Line EF shall also be * at Right Angles to a Plane passing thro' both GH and GK: But EF is parallel to AB; therefore AB is † also at Right Angles to the Plane passing thro' HGK. For the same Reason, CD is also at Right Angles to the Plane passing thro' HGK; and therefore AB, and CD, will be both at Right Angles to the Plane passing thro' HGK. But if two Right Lines be at Right Angles to the same Plane, they shall be * parallel to each other; therefore AB is parallel to CD. And so, *Right Lines that are parallel to the same Right Line, not being in the same Plane with it, are also parallel to each other*; which was to be demonstrated.

* 4 *of this.*
† 6 *of this.*
* 6 *of this.*

PROPOSITION X.

THEOREM.

If two Right Lines, touching one another, be parallel to two other Right Lines, touching one another, but not in the same Plane, those Right Lines contain equal Angles.

LET two Right Lines AB, BC, touching one another, be parallel to two Right Lines DE, EF, touching one another, but not in the same Plane. I say, the Angle ABC is equal to the Angle DEF.

For,

For, take BA, BC, ED, EF, equal one to another, and join AD, CF, BE, AC, DF: Then, becauſe BA is equal and parallel to ED, the Right Line AD ſhall alſo be * equal and parallel to BE. For the ſame Reaſon, CF will be equal and parallel to BE; therefore AD, CF, are both equal and parallel to BE. But Right Lines that are parallel to the ſame Right Line, not being in the ſame Plane with it, will be † parallel to each other. Therefore AD is parallel and equal to CF; but AC, DF, join them; wherefore AC is ‡ equal and parallel to DF. And becauſe the two Right Lines AB, BC, are equal to the two Right Lines DE, EF, and the Baſe AC equal to the Baſe DF; therefore the Angle ABC will be * equal to the Angle DEF. Whence, *if two Right Lines, touching one another, be parallel to two other Right Lines touching one another, but not in the ſame Plane, thoſe Right Lines contain equal Angles*; which was to be demonſtrated.

* 33. 1.
† 9 *of this.*
‡ 33. 1.
* 8. 1.

PROPOSITION XI.

PROBLEM.

From a Point given above a Plane, to draw a Right Line perpendicular to that Plane.

LET A be the Point given, above the given Plane BH. It is required to draw a Right Line from the Point A, perpendicular to the Plane BH.

Let a Right Line BC be any how drawn in the Plane BH; and let AD be drawn * from the Point A, perpendicular to BC; then if AD be perpendicular to the Plane BH, the Thing required is already done; but, if not, let DE be drawn in the Plane from the Point D, at Right Angles to BC; and let AF be drawn * from the Point A, perpendicular to DE: Laſtly, thro' F draw GH, parallel to BC.

* 12. 1.

Then, becauſe BC is perpendicular to both DA and DE, BC will alſo be † perpendicular to a Plane paſſing thro' ED, DA. But GH is parallel to BC; and if there are two Right Lines parallel, one of which is at Right Angles to ſome Plane, then ſhall the other be ‡ at Right Angles to the ſame Plane: Wherefore GH is at Right Angles to the Plane paſſing thro' ED, DA,

† 4 *of this.*
‡ 8 *of this.*

and

* Def. 3. and so is * perpendicular to all the Right Lines, in the same Plane that touch it. But AF, which is in the Plane passing thro' ED and DA, doth touch it. Therefore GH is perpendicular to AF; and so AF is perpendicular to GH; but AF, likewise, is perpendicular to DE; therefore AF is perpendicular to both HG, DE. But if, a Right Line stands at Right Angles to two Right Lines, in their common Section, that Line will † 4 of this. be † at Right Angles to the Plane passing thro' these Lines. Therefore AF is perpendicular to the Plane drawn thro' ED, GH; that is, to the given Plane BH. Therefore, AF *is drawn from the given Point* A, *above the given Point* BH, *perpendicular to the said Plane*; which was to be done.

PROPOSITION XII.

PROBLEM.

To erect a Right Line perpendicular to a given Plane, from a Point given therein.

LET A be a given Point in a given Plane MN. It is required to draw a Right Line from the Point A, at Right Angles to the Plane MN.

Let some Point B be supposed above the given Plane, * 11 of this. from which let BC be drawn * perpendicular to the † 31. 1. said Plane; and let AD be drawn † from A, parallel to BC.

Then, because AD, CB, are two parallel Right Lines, one of which, *viz.* BC, is perpendicular to the ‡ 8 of this. Plane MN; the other, AD, shall be ‡ also perpendicular to the same Plane. Therefore, *a Right Line is erected perpendicular to a given Plane, from a Point given thereon*; which was to be done.

PRO-

PROPOSITION XIII.

THEOREM.

Two Right Lines cannot be erected at Right Angles to a given Plane, from a Point therein given.

FOR, if it is poſſible, let two Right Lines AB, AC, be erected perpendicular to a given Plane on the ſame Side, at a given Point A, in the given Plane.

Then let a Plane be drawn thro' BA, AC, cutting the given Plane thro' A in the Right Line * DAE; but the Line DAE being in the given Plane, touches it; therefore the Right Lines AB, AC, DAE, are in one Plane: And becauſe CA is perpendicular to the given Plane, it ſhall alſo be † perpendicular to all Right Lines drawn in that Plane, and touching it: Therefore the Angle CAE is a Right Angle. For the ſame Reaſon, BAE is alſo a Right Angle; wherefore the Angle CAE is equal to BAE, and they are both in one Plane; which is abſurd. Therefore, *two Right Lines cannot be erected at Right Angles, to a given Plane, from a Point therein given*; which was to be demonſtrated.

* 3 *of this.*

† *Def.* 3.

PROPOSITION XIV.

THEOREM.

Thoſe Planes, to which the ſame Right Line is perpendicular, are parallel to each other.

LET the Right Line AB be perpendicular to each of the Planes DC, EF. I ſay, theſe Planes are parallel.

For, if they be not, let them be produced till they meet each other, and let the Right Line GH be the common Section, in which take any Point K, and AK, BK. Then, becauſe AB is perpendicular to the Plane EF, it ſhall alſo be perpendicular to the Right Line BK, being in the Plane EF produced; wherefore the Angle ABK is a Right Angle. And, for the ſame Reaſon, BAK is alſo a Right Angle.

And

And fo the two Angles ABK, BAK, of the Triangle
* 17. 1. ABK, are equal to two Right Angles, which is * impoffible: Therefore the Planes CD, EF, being produced, will not meet each other; and fo are neceffarily parallel. Therefore, *thofe Planes, to which the fame Right Line is perpendicular, are parallel to each other*; which was to be demonftrated.

PROPOSITION XV.

THEOREM.

If two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the fame Plane with them; the Planes drawn thro' thofe Right Lines are parallel to each other.

LET two Right Lines AB, BC, touching one another, be parallel to two Right Lines DE, EF, touching one another, but not in the fame Plane with them. I fay, the Planes paffing thro' AB, BC, and DE, EF, being produced, will not meet each other.

* 11 of this. For, let BG be drawn * from the Point B, perpendicular to the Plane paffing thro' DE, EF, meeting the fame in the Point G; and thro' G let GH be
‡ 31. 1. drawn ‡ parallel to ED, and GK parallel to EF; then, becaufe BG is perpendicular to the Plane paffing thro'
* Def. 3. DE, EF, it fhall alfo make * Right Angles with all Right Lines that touch it, and are in the fame Plane. But GH and GK, which are both in the fame Plane, touch it; therefore each of the Angles BGH, BGK, is a Right Angle. And fince BA is parallel to GH,
* 29. 1. the Angles GBA, BGH, are * equal to two Right Angles: But GA is a Right Angle; wherefore GBA fhall alfo be a Right Angle; and fo BG is perpendicular to BA. For the fame Reafon, GB is alfo perpendicular to BC; therefore fince a Right Line BG ftands at Right Angles to two Right Lines BA, BC,
‡ 4 of this. mutually cutting each other; BG fhall alfo be ‡ at Right Angles to the Plane drawn thro' BA, BC. But it is perpendicular to the Plane drawn thro' DE, EF; therefore BG is perpendicular to both the Planes drawn

thro'

thro' AB, BC, and DE, EF. But those Planes to which the same Right Line is perpendicular, are * parallel; therefore the Plane drawn thro' AB, BC, is parallel to the Plane drawn thro' DE, EF. Wherefore, *if two Right Lines, touching one another, be parallel to two Right Lines touching one another, and not being in the same Plane with them; the Planes drawn thro' those Right Lines are parallel to each other*; which was to be demonstrated. * 14 of this.

PROPOSITION XVI.

THEOREM.

If two parallel Planes are cut by another Plane, their common Sections will be parallel.

LET two parallel Planes AB, CD, be cut by any Plane EFGH; and let their common Sections be EF, GH. I say, EF is parallel to GH.

For, if it is not parallel, EF, GH, being produced, will meet each other either on the Side FH, or EG. First, let them be produced on the Side FH, and meet in K; then, because EFK is in the Plane AB, all Points taken in EFK will be in the same Plane. But K is one of the Points that is in EFK; therefore K is in the same Plane AB. For the same Reason K is also in the Plane CD; wherefore the Planes AB, CD, will meet each other. But they do not meet, since they are supposed parallel; therefore the Right Lines EF, GH, will not meet on the Side FH. After the same manner it is proved, that they will not meet, if produced, on the Side EG. But Right Lines, that will neither Way meet each other, are parallel; therefore EF, is parallel to GH. *If*, therefore, *two parallel Planes are cut by any other Plane, their common Sections will be parallel*; which was to be demonstrated.

PROPOSITION XVII.

THEOREM.

If two Right Lines are cut by parallel Planes, they shall be cut in the same Proportion.

LET two Right Lines AB, CD, be cut by parallel Planes GH, KL, MN, in the Points A, E, B, C, F, D. I say, as the Right Line AE is to the Right Line EB, so is CF to FD.

For, let AC, BD, AD, be joined; let AD meet the Plane KL in the Point X; and join EX, XF. Then, because two parallel Planes KL, MN, are cut by the
* 16 of this. Plane EBDX, their common Sections EX, BD, are * parallel. For the same Reason, because two parallel Planes GH, KL, are cut by the Plane AXFC, their common Sections AC, FX, are parallel; and because EX is drawn parallel to the Side BD of the Triangle
† 2. 6. ABD, it shall be, as AE is to EB, so is † AX to XD. Again, because XF is drawn parallel to the Side AC of the Triangle ADC, it shall be †, as AX is to XD, so is CF to FD. But it has been proved, as AX is to XD,
‡ 11. 5. so is AE to EB. Therefore, as AE is to EB, so is ‡ CF to FD. Wherefore, *if two Right Lines are cut by parallel Planes, they shall be cut in the same Proportion*; which was to be demonstrated.

PROPOSITION XVIII.

THEOREM.

If a Right Line be perpendicular to some Plane, then all Planes passing thro' that Line will be perpendicular to the same Plane.

LET the Right Line AB be perpendicular to the Plane CL. I say, all Planes that pass thro' AB, are likewise perpendicular to the Plane CL.

For, let a Plane DE pass thro' the Right Line AB, whose common Section, with the Plane CL, is the Right Line CE; and take some Point F in CE; from which let FG be drawn in the Plane DE, perpendicu-

lar

lar to the Right Line CE: Then, becauſe AB is perpendicular to the Plane CL, it ſhall alſo be * perpendicular to all the Right Lines which touch it, and are in the ſame Plane: Wherefore it is perpendicular to CE; and, conſequently, the Angle ABF is a Right Angle: But GFB is likewiſe a Right Angle; therefore AB is parallel to FG. But AB is at Right Angles to the Plane CL; therefore FG will be † at Right Angles to that ſame Plane. But one Plane is perpendicular to another, when the Right Lines drawn in one of the Planes, perpendicular to the common Section of the Planes, are ‡ perpendicular to the other Plane. But FG is drawn in one Plane DE, perpendicular to the common Section CE of the Planes, and it has been proved to be perpendicular to the Plane CL: In like manner any other Line in the Plane DE, drawn perpendicular to CE, is proved to be perpendicular to the Plane CL. Therefore the Plane DE is at Right Angles to the Plane CL. After the ſame manner we demonſtrated, that all Planes paſſing thro' the Right Line AB, are perpendicular to the Plane CL. Therefore, *if a Right Line be perpendicular to ſome Plane, then all Planes, paſſing thro' that Line, will be perpendicular to the ſame Plane*; which was to be demonſtrated.

* Def. 3.

† 8 of this.

‡ Def. 4. of this.

PROPOSITION XIX.

THEOREM.

If two Planes, cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plane.

LET two Planes AB, BC, cutting each other, be perpendicular to ſome third Plane, and let their common Section be BD. I ſay, BD is perpendicular to the ſaid third Plane, which let be ADC.

For, if poſſible, let BD not be perpendicular to the third Plane; and from the Point D let DE be drawn in the Plane AB, perpendicular to AD; and let DF be drawn, in the Plane BC, perpendicular to CD: Then, becauſe the Plane AB is perpendicular to the third Plane, and DE is drawn in the Plane AB, perpendicular to their common Section AD; DE

ſhall

* Def. 4. shall be * perpendicular to the third Plane ADC. In like manner we prove, that DF also is perpendicular to the said Plane; wherefore two Right Lines stand at Right Angles to this third Plane, on the same Side, at † 13 of this. the same Point D; which is † absurd: Therefore, to this third Plane cannot be erected any Right Lines perpendicular at D, and on the same Side, except BD, the common Section of the Planes AB, BC: Wherefore DB is perpendicular to the third Plane. *If* therefore, *two Planes, cutting each other, be perpendicular to some Plane, then their common Section will be perpendicular to that same Plane*; which was to be demonstrated.

PROPOSITION XX.

THEOREM.

If a solid Angle be contained under three plane Angles, any two of them, howsoever taken, are greater than the third.

LET the solid Angle A be contained under three plane Angles BAC, CAD, DAB. I say, any two of the Angles BAC, CAD, DAB, are greater than the third, howsoever taken.

For, if the Angles BAC, CAD, DAB, be equal, it is evident, that any two, howsoever taken, are greater than the third; but, if not, let BAC be the greater, * 23. 1. and make * the Angle BAE, at the Point A, with the Right Line AB, in a Plane passing thro' BA, AC, equal to the Angle DAB; make AE equal to AD; thro' E draw BEC, cutting the Right Lines AB, AC, in the Points B, C; and join DB, DC: Then, because DA is equal to AE, and AB is common, the two Sides DA, AB, are equal to the two Sides AE, AB; but the Angle DAB, is equal to the † 4. 1. Angle BAE; therefore the Base DB is † equal to the Base BE: And since the two Sides DB, DC, are greater than BC, and DB has been proved equal to BE; therefore the remaining Side DC shall be greater than the remaining Side EC; and since DA is equal to AE, and AC is common, and the Base DC greater ‡ 25. 1. than the Base EC; the Angle DAC shall be ‡ greater than

than the Angle EAC. But, from Construction, the Angle DAB, is equal to the Angle BAE; wherefore the Angles DAB, DAC, are greater than the Angle BAC. After this manner we demonstrate, if any two other Angles be taken, that they are greater than the third. Therefore, *if a solid Angle be contained under three plane Angles, any two of them, howsoever taken, are greater than the third*; which was to be demonstrated

PROPOSITION XXI.

Theorem.

Every solid Angle is contained under plane Angles, together, less than four Right ones.

LET A be a solid Angle, contained under plane Angles BAC, CAD, DAB. I say, the Angles BAC, CAD, DAB, are less than four Right Angles.

For, take any Points B, C, D, in each of the Lines AB, AC, AD; and join BC, CD, DB: Then, because the solid Angle at B is contained under three plane Angles CBA, ABD, CBD; any two of these are * greater than the third: Therefore the Angles CBA, ABD, are greater than the Angle CBD. For the same Reason, the Angles BCA, ACD, are greater than the Angle BCD; and the Angles CDA, ADB, greater than the Angle CDB. Wherefore the six Angles CBA, ABD, BCA, ACD, CDA, ADB, are greater than the three Angles CBD, BCD, CDB. But the three Angles CBD, BCD, CDB, are † equal to two Right Angles; wherefore the six Angles CBA, ABD, BCA, ADC, DCA, ADB, are greater than two Right Angles. And since the three Angles of each of the Triangles ABC, ACB, ADB, are equal to two Right Angles, the nine Angles of those Triangles CBA, BCA, BAC, ACD, CAD ADC, ADB, ABD, DAB, are equal to six Right Angles; six of which Angles CBA, BCA, ACD, ADC, ADB, ABD, are greater than two Right Angles. Therefore the three other Angles BAC, CAD, DAB, which contain the

* 20 of this.

† 32. 1.

the ſolid Angle, will be leſs than four Right Angles. Wherefore, *every ſolid Angle is contained under plane Angles, together, leſs than four Right ones*; which was to be demonſtrated.

PROPOSITION XXII.

THEOREM.

If there be three plane Angles, whereof two, any how taken, are greater than the third, and the Right Lines that contain them be equal; then it is poſſible to make a Triangle of the Right Lines joining the equal Right Lines which form the Angles.

LET ABC, DEF, GHK, be given plane Angles, any two whereof are greater than the third; and let the equal Right Lines AB, BC, DE, EF, GH, HK, contain them; and let AC, DF, GK, be joined. I ſay, it is poſſible to make a Triangle of AC, DF, GK; that is, any two of them, howſoever taken, are greater than the third.

For, if the Angles at B, E, H, are equal; then AC,
* 4. 1. DF, GK, will be * equal, and any two of them greater than the third; but, if not, let the Angles at B, E, H, be unequal; and let the Angle B be greater than either of the others at E, or H: Then the Right
† 24. 1. Line AC will be † greater than either DF, or GK; and it is manifeſt, that AC, together with either DF, or GK, is greater than the other. I ſay, likewiſe, that
‡ 23. 1. DF, GK, together, are greater than AC. For make ‡, at the Point B, with the Right Line AB, the Angle ABL equal to the Angle GHK; and make BL equal to either AB, BC, DE, EF, GH HK; and join AL, CL. Then, becauſe the two Sides AB, BL, are equal to the two Sides GH, HK, each to each; and they contain equal Angles; the Baſe AL ſhall be equal to the Baſe GK. And ſince the Angle E and H are greater than the Angle ABC, the Angle GHK is equal to the Angle ABL, and therefore the other Angle at E ſhall be greater than the Angle LBC. And ſince the two Sides LB, BC, are equal to the two Sides DE, EF, each to each, and the Angle DEF

DEF is greater than the Angle LBC, the Base DF shall be * greater than the Base LC But GK has * 24. 1.
been proved equal to AL; therefore DF, GK, are greater than KL, LC: But AL, LC, are greater than AC; wherefore DF, GK, shall be much greater than AC. Therefore, any two of the Right Lines AC, DF, GK, howsoever taken, are greater than the third: And so, *a Triangle may be made of* AC, DF, GK; which was to be demonstrated.

PROPOSITION XXIII.

PROBLEM.

To make a solid Angle of three plane Angles, whereof any two, howsoever taken, are greater than the third; but these three Angles must be less than four Right Angles.

LET ABC, DEF, GHK, be three plane Angles given, whereof any two, howsoever taken, are greater than the third; and let the said three Angles be less than four Right Angles; it is required to make a solid Angle of three plane Angles equal to ABC, DEF, GHK.

Let the Right Lines AB, BC, DE, EF, GH, HK, be cut off equal; and join AC, DF, GK; then it is possible to make * a Triangle of three Right Lines * 22 of this.
equal to AC, DF, GK: And so † let the Triangle † 22. 1.
LMN be made, so that AC be equal to LM, and DF to MN, and GK to LN; and let the Circle LMN be described ‡ about the Triangle, whose Centre let ‡ 5. 4.
be X, which will be either within the Triangle LMN, or on one Side thereof, or without the same.

First, let it be within, and join LK, MX, NX: I say, AB is greater than LX. For, if this be not so, AB shall be either equal to LX, or less. First, let it be equal; then, because AB is equal to LX, and also to BC, LX shall be equal to BC: But LX is equal to XM; therefore the two Sides AB, BC, are equal to the two Sides LX, XM, each to each; but the Base AC is put equal to the Base LM; wherefore the Angle ABC shall be * equal to the Angle LXM. * 8. 1.

For

For the same Reason, the Angle DEF is equal to the Angle MXN, and the Angle GHK to the Angle NXL; therefore the three Angles ABC, DEF, GHK, are equal to the three Angles LXM, MXN, NXL.
* Cor. 15. 1. But the three Angles LXM, MXN, NXL are * equal to four Right Angles; and so the three Angles ABC, DEF, GHK, shall also be equal to four Right Angles: But they are put less than four Right Angles, which is absurd; therefore AB is not equal to LX. I say also, it is neither less than LX; for, if this be possible, make XO equal to BA, and XP to BC, and join OP: Then, because AB is equal to BC, XO shall be equal to XP; and the remaining Part OL,
† 2. 6. equal to the remaining Part PM; and so LM is † parallel to OP, and the Triangle LMX is equiangular
‡ 4. 6. to the Triangle OPX: Wherefore XL is ‡ to LM, as XO is to OP; and (by Alternation) as XL is to XO, so is LM to OP. But LX is greater than XO; therefore LM shall also be greater than OP. But LM is put equal to AC; wherefore AC shall be greater than OP: And so, because the two Right Lines AB, BC, are equal to the two Right Lines OX, XP, and the Base AC greater than the Base OP; the Angle ABC
* 25. 1. will be * greater than the Angle OXP. In like manner we demonstrate, that the Angle DEF is greater than the Angle MXN, and the Angle GHK than the Angle NXL; therefore the three Angles ABC, DEF, GHK, are greater than the three Angles LXM, MXN, NXL: But the Angles ABC, DEF, GHK, are put less than four Right Angles; therefore the Angles LXM, MXN, NXL, shall be less by much than four Right Angles, and also
† Cor. 15. 1. equal † to four Right Angles; which is absurd: Wherefore AB is not less than LX. It has also been proved not to be equal to it; therefore it must
‡ 12 of this. necessarily be greater. On the Point X raise ‡ XR, perpendicular to the Plane of the Circle LMN, whose Length let be such, that the Square thereof be equal to the Excess by which the Square of AB exceeds the Square of LX; and let RL, RM, RN, be joined: Because RX is perpendicular to the Plane
* Def. 3. of the Circle LMN, it shall also be * perpendicular to LX, MX, NX: And because LX is equal to XM,

XM, and XR is common, and at Right Angles to them, the Base LR shall be* equal to the Base RM. * 4. 1.
For the same Reason, RN is equal RL, or RM; therefore three Right Lines RL, RM, RN, are equal to each other. And because the Square of XR is equal to the Excess by which the Square of AB exceeds the Square of LX, the Square of AB will be equal to the Squares of LX, XR, together: But the Square of RL is † equal to the Squares of LX, XR, for LXR is a † 47. 1.
Right Angle; therefore the Square of AB will be equal to the Square of RL; and so AB is equal to RL. But BC, DE, EF, GH, HK, are every one of them equal to AB; and RN, or RM, equal to RL; wherefore AB, BC, DE, EF, GH, HK, are each equal to RL, RM, or RN: And since the two Sides RL, RM, are equal to the two Sides AB, BC; and the Base LM is put equal to the Base AC; the Angle LRM shall be ‡ ‡ 8. 1.
equal to the Angle ABC. For the same Reason the Angle MRN is equal to the Angle DEF, and the Angle LRN equal to the Angle GHK: Therefore, *a solid Angle is made at* R *of three plane Angles* LRM, MRN, LRN, *equal to three plane Angles given* ABC, DEF, GHK.

Now, let the Centre of the Circle X be in one Side of the Triangle, *viz.* in the Side MN; and join XL. I say, again, that AB is greater than LX. For, if it be not so, AB will be either equal, or less than LX. First, let it be equal; then the two Sides AB, BC, are equal to the two Sides MX, LX, that is, they are equal to MN: But MN is put equal to DF; therefore DE, EF, are equal to DF, which is* impossible; * 20. 1.
therefore AB is not equal to LX. In like manner we prove, that it is neither lesser; for the Absurdity will much more evidently follow. Therefore AB is greater than LX. And if the Square of RX be made equal to the Excess by which the Square of AB exceeds the Square of LX, and RX be raised at Right Angles to the Plane of the Circle, *the Problem may be done* in like manner as before.

Lastly, Let the Centre X of the Circle be without the Triangle LMN, and join LX, MX, NX: I say, AB is greater than LX. For, if it be not, it must either be equal, or less. First, let it be equal; then

then the two Sides AB, BC, are equal to the two Sides MX, XL, each to each; and the Base AC is equal to the Base ML; therefore the Angle ABC is
† 8. 1. † equal to the Angle MXL. For the same Reason, the Angle GHK is equal to the Angle LXN; and so the whole Angle MXN is equal to the two Angles ABC, GHK: But the Angles ABC, GHK, are greater than the Angle DEF; therefore the Angle MXN is greater than DEF: But because the two Sides DE, EF, are equal to the two Sides MX, XN, and the Base DF is equal to the Base MN; the Angle MNX shall be † equal to the Angle DEF: But it has been proved greater, which is absurd; therefore AB is not equal to LX. Moreover, we will prove, that it is not less; wherefore it shall be necessarily greater. And if, again, XR be raised at Right Angles to the Plane of the Circle, and made equal to the Side of that Square by which the Square of AB exceeds the Square of LX; the Problem will be determined. Now, I say, AB is not less than LX: For, if it is possible that it can be less, make XO equal to AB, and XP equal to BC, and join OP; then, because AB is equal to BC, XO shall be equal to XP, and the remaining Part OL equal to the remaining Part PM;
* 2. 6. therefore LM is * parallel to PO, and the Triangle LMX equiangular to the Triangle PXO: Where-
† 4. 6. fore, as † XL is to LM, so is XO to OP; and (by Alternation) as LX is to XO, so is LM to OP: But LX is greater than XO; therefore LM is greater than OP; but LM is equal to AC; wherefore AC shall be greater than OP: And so, because the two Sides AB, BC, are equal to the two Sides OX, XP, each to each; and the Base AC is greater than the Base OP;
‡ 25. 1. the Angle ABC shall be ‡ greater than the Angle OXP. So, likewise, if XR be taken equal to XO, or XP, and OR be joined, we prove, that the Angle GHK is greater than the Angle OXR. At the Point X, with the Right Line LX, make the Angle LXS equal to the Angle ABC, and the Angle LXT equal to the Angle GHK, and XS, XT, each equal to XO, and join OS, OT, ST; then, because the two Sides AB, BC, are equal to the two Sides OX, XS, and the Angle ABC is equal to the Angle OXS, the Base

AC,

AC, that is, LM, ſhall be equal to the Baſe OS. For the ſame Reaſon, LN is alſo equal to OT: And ſince the two Sides ML, LN, are equal to the two Sides OS, OT; and the Angle MLN, or POR, is evidently greater than the Angle SOT; the Baſe MN ſhall be greater than the Baſe ST. But MN is equal to DF; therefore DF ſhall be greater than ST. Wherefore, becauſe the two Sides EF, DE, are equal to the two Sides SX, XT; and the Baſe DF is greater than the Baſe ST; the Angle DEF ſhall be greater than the Angle SXT. But the Angle SXT is equal to the Angles ABC, GHK; therefore the Angle DEF is greater than the Angles ABC, GHK: But it is alſo leſs (by *Hyp.*) which is abſurd; and conſequently, AB is not leſs than LX. And ſo, *a ſolid Angle may be made of three plane Angles that have the neceſſary Limitations*; which was to be done.

PROPOSITION XXIV.

THEOREM.

If a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof are equal Parallelograms.

LET the Solid CDGH be contained under parallel Planes AC, GF, BG, CE, FB, AE. I ſay, the oppoſite Planes thereof are equal Parallelograms.

For, becauſe the parallel Planes BG, CE, are cut by the Plane AC, their common Sections are * parallel; wherefore AB is parallel to CD. Again, becauſe the two parallel Planes BE, AE, are cut by the Plane AC, their common Sections are parallel; therefore AD is parallel to BC. But AB has been proved to be parallel to CD; wherefore AC ſhall be a Parallelogram. After the ſame manner we demonſtrate, that CE, FG, GB, BF, and AE, are Parallelograms. * 16 *of this.*

Let AH, DF, be joined: Then, becauſe AB is parallel to DC, and BH to CF; the Lines AB, BH, touching each other, ſhall be parallel to the Lines DC, CF, touching each other, and not being in the ſame Plane; wherefore they ſhall † contain equal Angles: And ſo the Angle ABH is equal to the Angle † 10 *of this.*

‡ 34. 1. DCF. And since the two Sides AB, BH, are ‡ equal to the two Sides DC, CF, and the Angle ABH equal to the Angle DCF; the Base AH shall be * equal to the Base DF, and the Triangle ABH equal to the Triangle DFC: And since the Parallelogram BG is † double to the Triangle ABH, and the Parallelogram CE to the Triangle DCF; the Parallelogram BG shall be equal to the Parallelogram CE. In like manner we demonstrate, that the Parallelogram AC is equal to the Parallelogram GF, and the Parallelogram AE equal to the Parallelogram BF. If, therefore, *a Solid be contained under six parallel Planes, the opposite Planes thereof are Parallelograms*; which was to be demonstrated.

* 4. 1.

† 41. 1.

Coroll. It follows, from what has been now demonstrated, that, if a Solid be contained under six parallel Planes, the opposite Planes thereof are similar and equal, because each of the Angles are equal, and the Sides about the equal Angles are proportional.

PROPOSITION XXV.

THEOREM.

If a solid Parallelepipedon be cut by a Plane, parallel to opposite Planes, then, as Base is to Base, so is Solid to Solid.

LET the solid Parallelepipedon ABCD be cut by a Plane YE, parallel to the opposite Planes RA, DH. I say, as the Base EF Φ A is to the Base EHCF, so is the Solid ABFY to the Solid EGCD.

For, let AH be both Ways produced, and make HM, MN, &c. equal to EH, and AK, KL, &c. equal to AE; and let the Parallelograms LO, K Φ, HX, MS, as likewise the Solids LP, KR, H Ω, MT, be compleated: Then because the Right Lines LK, KA, AE, are equal; the Parallelograms LO, K Φ, AF, shall be * also equal; as likewise the Parallelograms K Ξ, KB, AG: And, moreover, † the Parallelograms L Ψ, KP, AR, for they are opposite to one another. For the same Reasons, the Parallelograms EC, HX, MS, also, are equal to each other; as also the Paral-

* 35. 1.

† 24. of this.

Parallelograms HG, HI, IN; and ſo are the Parallelograms DH, MΩ, NT: Therefore three Planes of the Solid LP are equal to three Planes of the Solid KR, or AY, each to each; and the Planes oppoſite to theſe are equal to them: Therefore the three Solids LP, KR, RY, will be equal * to each other. For the ſame Reaſon, the three Solids ED, HΩ, MT, are equal to each other; therefore the Baſe LF is the ſame Multiple of the Baſe AF, as the Solid LY is of the Solid AY. For the ſame Reaſon, the Baſe NF is the ſame Multiple of the Baſe HF, as the Solid NY is of the Solid ED; and if the Baſe LF be equal to the Baſe NF, the Solid LY ſhall be equal to the Solid NY; and if the Baſe LF exceeds the Baſe NF, the Solid LY ſhall exceed the Solid NY; and, if it be leſs, leſs: Wherefore, becauſe there are four Magnitudes; *viz.* the two Baſes AF, FH, and the two Solids AY, ED, whoſe Equimultiples are taken, to wit, the Baſe LF, and the Solid LY; and the Baſe NF, and the Solid NY: And ſince it is proved, if the Baſe LF exceeds the Baſe NF, then the Solid LY will exceed the Solid NY; if equal, equal; and if leſs, leſs: Therefore as the Baſe AF is to the Baſe FH, ſo is * the Solid AY to the Solid ED. Wherefore, *if a ſolid Parallelepipedon be cut by a Plane, parallel to oppoſite Planes; then, as Baſe is to Baſe, ſo ſhall Solid be to Solid*; which was to be demonſtrated.

* Def. 10. of this.

* Def. 5 5.

PROPOSITION XXVI.

PROBLEM.

At a Right Line given, and at a Point given in it, to make a ſolid Angle equal to a ſolid Angle given.

LET AB be a Right Line given, A a given Point in it, and D a given ſolid Angle contained under the plane Angles EDC, EDF, FDC; it is required to make a ſolid Angle at the given Point A, in the given Right Line AB, equal to the given ſolid Angle D.

Aſſume any Point F in the Right Line DF, from which let FG be drawn * perpendicular to the Plane

paſſing

passing thro' ED, DC, meeting the said Plane in the
† 23. 1. Point G, and join DG; make † the Angles BAL, BAK, at the given Point A, with the Right Line AB, equal to the Angles EDC, EDG.

Lastly, Make AK equal to DG, and at the Point
* 12 *of this.* K erect ‡ HK, at Right Angles to the Plane passing thro' BAL; and make KH equal to GF; and join HA. I say, the solid Angle at A, which is contained under the three plane Angles BAL, BAH, HAL, is equal to the solid Angle at D, which is contained under the plane Angles EDC, EDF, FDC: For let the equal Right Lines AB, DE, be taken; and join HB, KB, FE, GE: Then, because FG is perpendicular
* *Def.* 3. *of this.* to the Plane passing thro' ED, DC, it shall be * perpendicular to all the Right Lines touching it, that are in the said Plane: Wherefore both the Angles FGD, FGE, are Right Angles. For the same Reason, both the Angles HKA, HKB, are Right Angles; and because the two Sides KA, AB, are equal to the two Sides GD, DE, each to each, and contain equal An-
† 4. 1. gles, the Base BK shall be † equal to the Base EG: But KH is also equal to GF; and they contain Right Angles; therefore, HB shall be † equal to FE. Again, because the two Sides AK, KH, are equal to the two Sides DG, GF, and they contain Right Angles; the Base AH shall be equal to the Base DF: But AB is equal to DE; therefore the two Sides HA, AB, are equal to the two Sides FD, DE. But the Base HB is equal to the Base FE; and so the Angle BAH will be
‡ 8. 1. ‡ equal to the Angle EDF: For the same Reason, the Angle HAL is equal to the Angle FDC: For since, if AL be taken equal to DC; and KL, HL, GC, FC, be joined; the whole Angle BAL is equal to the whole Angle EDC; and the Angle BAK, a Part of the one, is put equal to the Angle EDG, a Part of the other; the Angle KAL, remaining, will be equal to the Angle GDC remaining. And because the two Sides KA, AL, are equal to the two Sides GD, DC, and they contain equal Angles; the Base KL will be equal to the Base GC: But KH is equal to GF; wherefore the two Sides LK, KH, are equal to the two Sides CG, GF: But they contain Right Angles; therefore the Base HL will be equal to the Base

FC.

FC. Again, because the two Sides HA, AL, are equal to the two Sides FD, DC; and the Base HL is equal to the Base FC; the Angle HAL will be equal to the Angle FDC: But the Angle BAL was made equal to the Angle EDC: Therefore, *a solid Angle is made equal to a solid Angle given*; which was to be done.

PROPOSITION XXVII.

PROBLEM.

Upon a Right Line given, to describe a Parallelepipedon, similar, and in like manner situate, to a solid Parallelepipedon.

LET AB be a Right Line, and CD a given solid Parallelepipedon; it is required to describe a solid Parallelepipedon upon the given Right Line, AB, similar, and alike situate, to the given solid Parallelepipedon CD.

Make a solid Angle at the given Point A, in the Right Line AB, * contained under the Angles BAH, HAK, KAB; so that the Angle BAH may be equal to the Angle ECF, the Angle BAK to the Angle ECG, and the Angle HAK to the Angle GCF; and make, as EC is to CG, so BA † to AK; and as GC to CF, so KA to AH: Then (by Equality of Proportion) as EC is to CF, so shall BA be to AH: Compleat the Parallelogram BH, and the Solid AL; then, because it is, as EC is to GC, so is AB to AK; *viz.* the Sides about the equal Angles ECG, BAK, proportional; the Parallelogram KB shall be similar to the Parallelogram GE. Also, for the same Reason, the Parallelogram KH shall be similar to the Parallelogram GF, and the Parallelogram HB to the Parallelogram FE: Therefore three Parallelograms of the Solid AL, are similar to three Parallelograms of the Solid CD. But these three Parallelograms are ‡ equal and similar to their three opposite ones; therefore the whole Solid AL will be similar to the whole Solid CD; and so, *a solid Parallelepipedon* AL *is described upon the given Right Line* AB, *similar and alike situate, to the given solid Parallelepipedon* CD; which was to be done.

* 26 *of this.*

† 12. 6.

‡ *Cor.* 20. *of this.*

PROPOSITION XXVIII.

THEOREM.

If a solid Parallelepipedon be cut by a Plane passing thro' the Diagonals of two opposite Planes, that Solid will be bisected by the Plane.

LET the solid Parallelepipedon AB be cut by the Plane CDEF, passing thro' the Diagonals CF, DE, of two opposite Planes. I say, the Solid AB is bisected by the Plane CDEF.

* 34. 1. For, because the Triangle CGF is * equal to the Triangle CBF, and the Triangle ADE to the Triangle
† 24 of this. DEH, and the Parallelogram CA to † the Parallelogram BE, for it is opposite to it; and the Parallelogram GE to the Parallelogram CH; the Prism contained by the two Triangles CGF, ADE, and the three Parallelograms GE, AC, CE, is equal to the Prism contained under the two Triangles CFB, DEH,
‡ Def. 10. of this. and the three Parallelograms CH, BE, CE; for ‡ they are contained under Planes equal in Number and Magnitude. Therefore, *the whole Solid* AB *is bisected by the Plane* CDEF; which was to be demonstrated.

PROPOSITION XXIX.

THEOREM.

Solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same Right Lines, are equal to one another.

LET the solid Parallelepipedons CM, BF, be constituted upon the same Base AB, with the same Altitude, whose insistent Lines AF, AG, LM, LN, CD, CE, BH, BK, are in the same Right Lines FN, DK. I say, the Solid CM is equal to the Solid BF.

For, because CH, CK, are both Parallelograms,
* 34. 1. CB shall be * equal to DH, or EK; wherefore DH

is equal to EK. Let EH, which is common, be taken away, then the Remainder DE will be equal to the Remainder HK, and so the Triangle DEC is † equal to the Triangle HKB, and the Parallelogram DG equal to the Parallelogram HN; for the same Reason the Triangle AFG is equal to the Triangle LMN. Now the Parallelogram CF ‡ is equal to the Parallelogram BM, and the Parallelogram CG to the Parallelogram BN, for they are opposite. Therefore the Prism contained under the two Triangles AFG, DEC, and the three Parallelograms CF, DG, CG, is * equal to the Prism contained under the two Triangles LMN, HBK, and the three Parallelograms BM, HN, BN. Let the common Solid, whose Base is the Parallelogram AB, opposite to the Parallelogram GEHM, be added, then the whole solid Parallelepipedon CM is equal to the whole solid Parallelepipedon BF. Therefore, *solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same Right Lines, are equal to one another*; which was to be demonstrated.

† ‡ * Def. 10. of this.

PROPOSITION XXX.

THEOREM.

Solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same Right Lines, are equal to one another.

LET there be solid Parallelepipedons CM, CN, having equal Altitudes, and standing on the same Base AB, and whose insistent Lines AF, AG, LM, LN, CD, CE, BH, BK, are not in the same Right Lines. I say, the Solid CM is equal to the Solid CN.

For let NK, DH, be produced, and GE, FM, be drawn, meeting each other in the Points R, X: Let also FM, GE, be produced to the Points O, P, and join AX, LO, CP, BR. The Solid CM, whose Base is the Parallelogam ACBL, being opposite to the Parallelogram FDHM, is * equal to the Solid CO, whose

* 29 of this.

whose Base is the Parallelogram ACBL, being opposite to XPRO, for they stand upon the same Base ACBL; and the insistent Lines AF, AX, LM, LO,
* CD, CP, BH, BR, are in the same right Lines FO, DR: But the Solid CO, whose Base is the Parallelo-
* 29 of this. gram ACBL, being opposite to XPRO, is * equal to the Solid CN, whose Base is the Parallelogram ACBL, being opposite to GEKN; for they stand upon the same Base ACBL, and their insistent Lines AG, AX, CE, CP, LN, LO, BK, BR, are in the same Right Lines GP, NR: Wherefore the Solid CM shall be equal to the Solid CN. Therefore, *solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same Right Lines, are equal to one another*; which was to be demonstrated.

PROPOSITION XXXI.

THEOREM.

Solid Parallelepipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another.

LET AE, CF, be solid Parallelepipedons, constituted upon the equal Bases AB, CD, and having the same Altitude. I say, the Solid AE is equal to the Solid CF.

First, Let HK, BE, AG, LM, OP, DF, Cε, RS, be at Right Angles to the Bases AB, CD; let the Angle ALB not be equal to the Angle CRD, and produce CR to T, so that RT be equal to AL; then make the Angle TRY, at the Point R, in the Right
* 23. 1. Line RT, equal * to the Angle ALB; make RY equal to LB; draw XY, thro' the Point Y, * parallel
‡ 31. 1. to RT ‡, and compleat the Parallelogram RX, and the Solid ΨY. Therefore, because the two Sides TR, RY, are equal to the two Sides AL, LB, and they contain equal Angles; the Parallelogram RX shall be equal and similar to the Parallelogram HL. And again, because AL is equal to RT, and LM to RS, and they contain equal Angles, the Parallelogram RΨ shall

ſhall be equal and ſimilar to the Parallelogram AM. For the ſame Reaſon the Parallelogram LE is equal and ſimilar to the Parallelogram SY; therefore three Parallelograms of the Solid AE are equal and ſimilar to three Parallelograms of the Solid ΨY; and ſo the three oppoſite ones of one Solid are † alſo equal and ſimilar to the three oppoſite ones of the other: Therefore the whole ſolid Parallelepipedon AE is equal to the whole ſolid Parallelepipedon ΨY. Let DR, XY, be produced, and meet each other in the Point Ω, and let TQ be drawn thro' T * parallel to DΩ, and produce TQ, OD, till they meet in V, and compleat the Solids ΩΨ, RI: Then the Solid ΨΩ, whoſe Baſe is the Parallelogram RΨ, and ΩΓ is that oppoſite to it, is ‡ equal to the Solid ΨY, whoſe Baſe is the Parallelogram RΨ, and YΦ is that oppoſite to it, for they ſtand upon the ſame Baſe RΨ, have the ſame Altitude, and their inſiſtent Lines RΩ, RY, TQ, TX, SZ, SN, ΨΓ, ΨΦ, are in the ſame Right Lines ΩX, ZΦ; but the Solid ΨY is equal to the Solid AE; and ſo AE is equal to the Solid ΨΩ. Again, becauſe the Parallelogram RYXT is equal to the Parallelogram ΩT, for it ſtands on the ſame Baſe RT, and between the ſame Parallels RT, ΩX; and the Parallelogram RYXT is equal to the Parallelogram CD, becauſe it is alſo equal to AB; the Parallelogram ΩT is equal to the Parallelogram CD, and DT is ſome other Parallelogram: Therefore, as the Baſe CD is to the Baſe DT, ſo is ΩT to TD. And becauſe the ſolid Parallelepipedon CI is cut by the Plane RF, being parallel to two oppoſite Planes; it ſhall be *, as the Baſe CD is to the Baſe DT, ſo is the Solid CF to the Solid RI. For the ſame Reaſon, becauſe the ſolid Parallelepipedon ΩI is cut by the Plane RΨ, parallel to two oppoſite Planes; as the Baſe ΩT is to the Baſe DT, ſo ſhall * the Solid ΩΨ be to the Solid RI. But as the Baſe CD is to the Baſe DT, ſo is the Baſe ΩT to TD: Therefore, as the Solid CF is to the Solid RI, ſo is the Solid ΩΨ to the Solid RI. And ſince each of the Solids CF, ΩΨ, has the ſame Proportion to the Solid RI, the Solid CF is equal to the Solid ΩΨ: But the Solid ΩΨ has been proved equal to the Solid AE; therefore

† 24 of this.

* 31. 1.

‡ 29 of this.

* 25 of this.

† 9. 5. fore the Solid AE shall be † equal to the Solid CF. But, now let the insistent Lines AG, HK, BE, LM, CN, OP, DF, RS, not be at Right Angles to the Bases AB, CD. I say, again, that the Solid AE is equal to the Solid CF. Let there be drawn from the Points K, E, G, M, P, F, N, S, to the Plane wherein are the Bases AB, CD, the Perpendiculars KΞ, ET, GY, MΦ, SI, FΨ, NΩ, PX, meeting the Plane in the Points Ξ, T, Y, Φ, I, Ψ, Ω, X; and join ΞT, YΦ, ΞY, TΦ, XΨ, XΩ, ΩI, ΨI; then the Solid KΦ is equal to the Solid PI, for they stand on equal Bases KM, PS, have the same Altitude, and the insistent Lines are at Right Angles to the Bases. But the Solid KΦ, is equal to the Solid AE, and the

‡ 29 of this. Solid PI to ‡ the Solid CF, since they stand upon the same Base, have the same Altitude, and their insistent Lines are in the same Right Line: Therefore the Solid AE shall be equal to the Solid CF. Wherefore, *solid Parallelepipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another*; which was to be demonstrated.

PROPOSITION XXXII.

THEOREM.

Solid Parallelepipedons, that have the same Altitude, are to each other as their Bases.

LET AB, CD, be solid Parallelepipedons, that have the same Altitude. I say, they are to one another as their Bases; that is, as the Base AE is to the Base CF, so is the Solid AB to the Solid CD.

For, apply a Parallelogram FH to the Right Line FG, equal to the Parallelogram AE; and compleat the solid Parallelepipedon GK upon the Base FH, having the same Altitude as CD has: Then the

* 31 of this. Solid AB is * equal to the Solid GK, for they stand upon equal Bases AE, FH, and have the same Altitude; and so, because the solid Parallelepipedon CK is cut by the Plane DG, parallel to two opposite

† 25 of this. Planes, it shall be †, as the Base HF is to the Base FC, so is the Solid HD to the Solid DC: But the Base

Base FH is equal to the Base AE, and the Solid AB to the Solid GK. Therefore, as the Base AE is to the Base CF, so is the Solid AB to the Solid CD. Wherefore, *solid Parallelepipedons, that have the same Altitude, are to each other as their Bases*; which was to be demonstrated.

PROPOSITION XXXIII.

THEOREM.

Similar solid Parallelepipedons are to one another in the triplicate Proportion of their homologous Sides.

LET AB, CD, be similar solid Parallelepipedons, and let the Side AE be homologous to the Side CF. I say, the Solid AB, to the Solid CD, hath a Proportion, triplicate of that, which the Side AE has to the Side CF.

For, produce AE, GE, HE, to EK, EL, EM; and make EK equal to CF, and EL to FN, and EM to FR; and let the Parallelogram KL, and likewise the Solid KO, be compleated: Then, because the two Sides KE, EL, are equal to the two Sides CF, FN; and the Angle KEL equal to the Angle CFN (since the Angle AEG is equal to the Angle CFN, because of the Similarity of the Solids AB, CD) the Parallelogram KL shall be similar and equal to the Parallelogram CN. For the same Reason, the Parallelogram KM is equal and similar to the Parallelogram CR, and the Parallelogram OE to DF; therefore three Parallelograms of the Solid KO are equal and similar to three Parallelograms of the Solid CD: But those three Parallelograms are * equal and similar to the three opposite Parallelograms; therefore the whole Solid KO is equal and similar to the whole Solid CD. Let the Parallelogram GK be compleated, as also the Solids EX, LP, upon the Bases GK, KL, having the same Altitude as AB: And since, because of the Similarity of the Solids AB and CD, it is, as AE is to CF, so is EG to FN; and so EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; it shall be, as

* 24 of this.

† 1. 6. AE is to EK, ſo is † the Parallelogram AG to the Parallelogram GK; but as GE is to EL, ſo is GK to KL; and as HE is † to EM, ſo is PE to KM: Therefore, as the Parallelogram AG is to the Parallelogram GK, ſo is GK to KL, and PE to KM. But as AG
‡ 32 *of this.* is to GK, ſo is ‡ the Solid AB to the Solid EX; and as GK is to KL, ſo is the Solid EX to the Solid PL; and as PE is to KM, ſo is the Solid PL to the Solid KO: Therefore, as the Solid AB is to the Solid EX,
* 11. 5. ſo is * EX to PL, and PL to KO: But if four Magnitudes be continually proportional, the firſt to the
† *Def.* 11. 5. fourth hath † a triplicate Proportion of that which it has to the ſecond. Therefore, alſo, the Solid AB to the Solid KO, hath a triplicate Proportion of that which AB has to EX: But as AB is to EX, ſo is the Parallelogram AG to the Parallelogram GK; and ſo is the Right Line AE to the Right Line EK: Wherefore the Solid AB to the Solid KO, hath a Proportion triplicate of that which AE has to EK. But the Solid KO is equal to the Solid CD, and the Right Line EK equal to the Right Line CF: Therefore, *the Solid* AB, *to the Solid* CD, *has a Proportion triplicate of that which the homologous Side* AE *has to the homologous Side* CF; which was to be demonſtrated.

Coroll. From hence it is manifeſt, if four Right Lines be continually proportional, as the firſt is to the fourth, ſo is a ſolid Parallelepipedon deſcribed upon the firſt, to a ſimilar ſolid Parallelepipedon, alike ſituate, deſcribed upon the ſecond; becauſe the firſt to the fourth, has a Proportion triplicate of that which it has to the ſecond.

PRO-

PROPOSITION XXXIV.

THEOREM.

The Bases and Altitudes of equal solid Parallelepipedons are reciprocally proportional; and those solid Parallelepipedons, whose Bases and Altitudes are reciprocally proportional, are equal.

LET AB, CD, be equal solid Parallelepipedons. I say, their Bases and Altitudes are reciprocally proportional; that is, as the Base EH is to the Base FP, so is the Altitude of the Solid CD to the Altitude of the Solid AB.

First, let the insistent Lines AG, EF, LB, HK, CM, NX, OD, PR, be at Right Angles to their Bases: I say, as the Base EH is to the Base NP, so is CM to AG. For, if the Base EH be equal to the Base NP, and the Solid AB is equal to the Solid CD; the Altitude CM shall also be equal to the Altitude AG: For if, when the Bases EH, NP, are equal, the Altitudes AG, CM, are not so; then the Solid AB will not be equal to the solid CD, but it is put equal to it: Therefore the Altitude CM is not unequal to the Altitude AG, and so they are necessarily equal to one another; and, consequently, as the Base EH is to the Base NP, so shall CM be to AG. But now let the Base EH be unequal to the Base NP, and let EH be the greater; then, since the Solid AB is equal to the Solid CD, CM is greater than AG; for, otherwise, it would follow, that the Solids AB, CD, are not equal, which are put such: Therefore, make CT equal to AG, and compleat the solid Parallelepipedon VC upon the Base NP having the Altitude CT Then, because the Solid AB is equal to the Solid CD, and VC is some other Solid; and since equal Magnitudes have * the same Proportion to the same Magnitudes; it shall be, as the Solid AB is to the Solid CV, so is the Solid CD to the Solid CV: But as the Solid AB is to the Solid CV, so is † the Base EH to the Base NP; for AB, CV, are Solids having equal Altitudes: And as the Solid CD is to the Solid CV, so

* 7. 5.

† 32 of this.

‡ 25 *of this.* is ‡ the Base MP to the Base PT, and so is MC to CT: Therefore, as the Base EH is to the Base NP, so is MC to CT. But CT is equal to AG; wherefore, as the Base EH is to the Base NP, so is MC to AG: Therefore, *the Bases and Altitudes of the equal solid Parallelepipedons* AB, CD, *are reciprocally proportional.*

Now, let the Bases and Altitudes of the solid Parallelepipedons AB, CD, be reciprocally proportional; that is, let the Base EH be to the Base NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB. I say, the Solid AB is equal to the Solid CD.

For, let again the insistent Lines be at Right Angles to the Bases; then, if the Base EH be equal to the Base NP, and EH is to NP as the Altitude of the Solid CD is to the Altitude of the Solid AB; the Altitude of the Solid CD shall be equal to the Altitude of the Solid AB. But solid Parallelepipedons, that stand
* 13 *of this.* upon equal Bases, and have the same Altitude, are * equal to each other; therefore the Solid AB is equal to the solid CD.

But now let the Base EH not be equal to the Base NP, and let EH be the greater; then the Altitude of the Solid CD is greater than the Altitude of the Solid AB; that is, CM is greater than AG: Again, put CT equal to AG, and compleat the Solid CV, as before; and then, because the Base EH is to the Base NP, as MC is to AG, and AG is equal to CT; it shall be, as the Base EH is to the Base NP, so is MC to CT: But as the Base EH is to the Base NP, so is the Solid AB to the Solid CV; for the Solids AB, CV, have equal Altitudes; and as MC is to CT, so is the Base MP to the Base PT, and so the Solid CD to the Solid CV: Therefore as the Solid AB is to the Solid CV, so is the Solid CD to the Solid CV: But since each of the Solids AB, CD, has the same Proportion to CV; the Solid AB shall be equal to the Solid CD; whence, *the two solid Parallelepipedons* AB, CD, *whose Bases and Altitudes are reciprocally proportional, are equal*; which was to be demonstrated.

Now, let the insistent Lines FE, BL, GA, KH, XN, DO, MC, RP, not be at Right Angles to the Bases; and from the Points F, G, B, K, X, M, D, R, let there be drawn Perpendiculars to the Planes of

of the Bases EH, NP, meeting the same in the Points S, T, Y, V, Q, Z, Ω, Φ, and compleat the Solids FV, X Ω. Then, I say, if the Solids AB, CD, be equal, their Bases and Altitudes are reciprocally proportional; *viz.* as the Base EH is to the Base NP, so is the Altitude of the Solid CD to the Altitude of the Solid AB.

For, because the Solid AB is equal to the Solid CD, and the Solid AB is * equal to the Solid BT; for they stand upon the same Base FK, and have the same Altitude; and the Solid DC is * equal to the Solid DZ, since they stand upon the same Base XR, and have the same Altitude; therefore the Solid BT shall be equal to the Solid DZ. But the Bases and Altitudes of those equal Solids, whose Altitudes are at Right Angles to their Bases, are † reciprocally proportional; therefore as the Base FK is to the Base XR, so is the Altitude of the Solid DZ to the Altitude of the Solid BT. But the Base FK is equal to the Base EH, and the Base XR to the Base NP; wherefore, as the Base EH is to the Base NP, so is the Altitude of the Solid DZ to the Altitude of the Solid BT. But the Solids DZ, DC, have the same Altitude, and so have the Solids BT, BA; therefore the Base EH is to the Base NP, as the Altitude of the Solid DC is to the Altitude of the Solid AB; and so, *the Bases and Altitudes of equal solid Parallelepipedons are reciprocally proportional.*

* 30 *of this.*

† *From what has been before proved.*

Again, let the Bases and Altitudes of the solid Parallelepipedons AB, CD, be reciprocally proportional; *viz.* as the Base EH is to the Base NP, so let the Altitude of the Solid CD be to the Altitude of the Solid AB: I say, the Solid AB is equal to the Solid CD.

For, the same Construction remaining, because the Base EH is to the Base NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; and since the Base EH is equal to the Base FK, and NP to XR; it shall be, as the Base FK is to the Base XR, so is the Altitude of the solid CD to the Altitude of the Solid AB. But the Altitudes of the Solids AB, BT, are the same; as also of the Solids CD, DZ; therefore the Base FK is to the Base XR, as the Altitude of the Solid DZ is to the Altitude of the Solid BT; wherefore the Bases and Altitudes of the solid Parallelepipe-

dons BT, DZ, are reciprocally proportional; but those solid Parallelepipedons, whose Altitudes are at Right Angles to their Bases, and the Bases and Altitudes are reciprocally proportional, are equal to † each other. But the Solid BT is equal to the Solid BA, for they stand upon the same Base FK, and have the same Altitude; and the Solid DZ is also equal to the Solid DC, since they stand upon the same Base XR, and have the same Altitude: Therefore the Solid AB is equal to the Solid CD; whence *solid Parallelepipedons, whose Bases and Altitudes are reciprocally proportional, are equal*; which was to be demonstrated.

† *From what has been before proved.*

PROPOSITION XXXV.

THEOREM.

If there be two plane Angles equal, and from the Vertices of those Angles two Right Lines be elevated above the Planes, in which the Angles are, containing equal Angles with the Lines first given, each to its correspondent one; and if in those elevated Lines any Points be taken, from which Lines be drawn perpendicular to the Planes in which the Angles first given are, and Right Lines be drawn to the Angles first given from the Points made by the Perpendiculars in the Planes; those Right Lines will contain equal Angles with the elevated Lines.

LET BAC, EDF, be two equal Right-lined plane Angles, and from A and D, the Vertices of those Angles, let two Right Lines, AG and DM, be elevated above the Planes of the said Angles, making equal Angles with the Lines first given, each to its correspondent one; *viz.* the Angle MDE equal to the Angle GAB, and the Angle MDF to the Angle GAC; and take any Points G and M in the Right Lines AG, DM; from which let GL and MN be drawn perpendicular to the Planes passing thro' BAC, EDF, meeting the same in the Points L and N; and join LA and ND. I say, the Angle GAL is equal to the Angle MDN.

Make

Make AH equal to DM, and thro' H let HK be drawn parallel to GL; but GL is perpendicular to the Plane paſſing thro' BAC; therefore HK ſhall be † also perpendicular to the Plane paſſing thro' BAC: Draw from the Points K and N, to the Right Lines AB, AC, DE, and DF, the Perpendiculars KB, KC, NE, NF; and join HC, CB, MF, FE: Then, because the Square of HA is ‡ equal to the Squares of HK, KA; and the Squares of KC and CA are † equal to the Square of KA; the Square of HA ſhall be equal to the Squares of HK, KC, and CA: But the Square of HC is equal to the Squares of HK and KC; therefore the Square of HA will be equal to the Squares of HC and CA; and ſo the Angle HCA is ‡ a Right Angle. For the ſame Reaſon, the Angle DFM is alſo a Right Angle; therefore the Angle ACH is equal to DFM: But the Angle HAC is alſo equal to the Angle MDF; therefore the two Triangles MDF, HAC, have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other; *viz.* that which is ſubtended by one of the equal Angles; that is, the Side HA equal to DM; and ſo the other Sides of the one ſhall be * equal to the other Sides of the other, each to each: Wherefore AC is equal to DF. In like manner we demonſtrate, that AB is equal to DE: For, let HB, ME, be joined; then, becauſe the Square of AH is equal to the Squares of AK and KH; and the Squares of AB and BK are equal to the Square of AK; the Squares of AB, BK, and KH, will be equal to the Square of AH. But the Square of BH is equal to the Squares of BK, KH; for the Angle HKB is a Right Angle, becauſe HK is perpendicular to the Plane paſſing thro' BAC; therefore the Square of AH is equal to the Squares of AB and BH: Wherefore the Angle ABH is † a Right Angle. For the ſame Reaſon the Angle DEM is alſo a Right Angle; and the Angle BAH is equal to the Angle EDM, for ſo it is put; and AH is equal to DM; therefore AB is * alſo equal to DE: And ſo, ſince AC is equal to DF, and AB to DE; the two Sides CA, AB, ſhall be equal to the two Sides FD, DE: But the Angle BAC is equal to the Angle FDE; therefore the Baſe BC is * equal to the Baſe EF, the Triangle to the Triangle, and the other

† 8 *of this.*
‡ 47. 1.
‡ 48. 1.
* 26. 1.
† 48. 1.
* 4. 1.

other Angles to the other Angles: Wherefore the Angle ACB is equal to the Angle DFE. But the Right Angle ACK is equal to the Right Angle DFN; and therefore the remaining Angle BCK is equal to the remaining Angle EFN. For the same Reason, the Angle CBK is equal to the Angle FEN; and so, because BCK and EFN are two Triangles, having two Angles equal to two Angles, each to each, and one Side equal to one Side, which is at the equal Angles; *viz.* BC
* 26. 1. equal to EF; therefore * they shall have the other Sides equal to the other Sides: Therefore CK is equal to FN. But AC is equal to DF; therefore the two Sides AC, CK, are equal to the two Sides DF, FN; and they contain Right Angles; consequently, * the Base AK is equal to the Base DN. And since AH is equal to DM, the Square of AH shall be equal to the Square of DM: But the Squares of AK and KH are equal to the Square of AH; for the Angle AKH is a Right Angle; and the Squares of DN and NM are equal to the Square of DM, since the Angle DNM is a Right Angle; therefore the Squares of AK and KH are equal to the Squares of DN and NM; of which the Square of AK is equal to the Square of DN: Wherefore the Square of KH remaining is equal to the remaining Square of NM; and so the Right Line HK is equal to MN. And since the two Sides HA, AK, are equal to the two Sides MD, DN, each to each, and the Base KH has been proved equal to the Base NM,
† 8. the Angle HAK, or GAL, shall be † equal to the Angle MDN; *which was to be demonstrated.*

Coroll. From hence it is manifest, that if there be two Right-lined plane Angles equal, from whose Points equal Right Lines be elevated on the Planes of the Angles, containing equal Angles with the Lines first given, each to each; Perpendiculars drawn from the extreme Points of those elevated Lines to the Planes of the Angles first given, are equal to one another.

PRO-

PROPOSITION XXXVI.

THEOREM.

If three Right Lines be proportional, the solid Parallelepipedon made of them is equal to the solid Parallelepipedon made of the middle Line, if it be an equilateral one, and equiangular to the aforesaid Parallelepipedon.

LET three Right Lines A, B, C, be proportional; *viz.* let A be to B, as B is to C. I say, the Solid made of A, B, C, is equal to the equilateral Solid made of B, equiangular to that made on A, B, C.

Let E be a solid Angle contained under the three plane Angles DEF, GEF, FED; and make DE, GE, EF, each equal to B, and compleat the solid Parallelepipedon EK: Again, put LM equal to A, and at the Point L, at the Right Line LM, make * a solid Angle contained under the plane Angles NLX, XLM, MLN, equal to the solid Angle E; and make LN equal to B, and LX equal to C: Then, because A is to B, as B is to C; and A is equal to LM; and B to LN, EF, EG, or ED; and C to LX; it shall be, as LM is to EF, so is GE to LX: And so the Sides about the equal Angles MLX, GEF, are reciprocally proportional. Wherefore the Parallelogram MX † is equal to the Parallelogram GF. And since the two plane Angles GEF, XLM, are equal, and the Right Lines LN, ED, being equal, are erected at the angular Points containing equal Angles with the Lines first given, each to each; the Perpendiculars drawn ‡ from the Points N and D, to the Planes drawn thro' XLM, GEF, are equal one to another: Therefore the Solids LH, EK, have the same Altitude. But solid Parallelepipedons that have equal Bases, and the same Altitude, are * equal to each other; therefore the Solid HL is equal to the Solid EK. But the Solid HL is that made of the three Right Lines A, B, C; and the Solid EK, that made of the Right Line B: Therefore, *if three Right Lines be proportional, the solid Parallelepipedon made of them is equal to the solid Parallelepipedon made of the middle Line, if it be an equilateral one,* and

* 26 of this.

† 14. 6.

‡ Cor. 35. of this.

* 31 of this.

and equiangular to the aforesaid Parallelepipedon; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If four Right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar, and alike described, be proportional, then the Right Lines they are described from, shall be proportional.

LET the four Right Lines AB, CD, EF, GH, be proportional; *viz.* let AB be to CD, as EF is to GH; and let the similar and alike situate Parallelepipedons KA, LC, ME, NG, be described from them. I say, KA is to LC, as ME is to NG.

For, because the solid Parallelepipedon KA is similar to LC, therefore KA to LC shall be * a Proportion triplicate of that which AB has to CD. For the same Reason, the Solid ME to NG will have a triplicate Proportion of that which EF has to GH. But AB is to CD, as EF is to GH; therefore AK is to LC, as ME is to NG. And if the Solid AK be to the Solid LC, as the Solid ME is to the Solid NG; I say, as the Right Line AB is to the Right Line CD, so is the Right Line EF to the Right Line GH: For, because AK to LC has † a Proportion triplicate of that which AB has to CD; and ME to NG has a Proportion triplicate of that which EF has to GH; and since AK is to LC, as ME is to NG; it shall be, as AB is to CD, so is EF to GH. Therefore, *if four Right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar and alike described, be proportional, then the Right Lines they are described from, shall be proportional*; which was to be demonstrated.

* 33 *of this.*

† 33 *of this.*

PRO-

PROPOSITION XXXVIII.

THEOREM.

If a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular shall fall in the common Section of the Planes.

LET the Plane CD be perpendicular to the Plane AB, let their common Section be AD, and let some Point E be taken in the Plane CD. I say, a Perpendicular, drawn from the Point E to the Plane AB, falls on AD.

For, if it does not, let it fall without the same, as EF, meeting the Plane AB in the Point F; and from the Point F let FG be drawn in the Plane AB, perpendicular to AD; this shall be * perpendicular to the Plane CD; and join EG: Then, because FG is perpendicular to the Plane CD, and the Right Line EG, in the Plane CD, touches it; the Angle FGE shall be † a Right Angle. But EF is also at Right Angles to the Plane AB; therefore the Angle EFG is a Right Angle: And so, two Angles of the Triangle EFG are equal to two Right Angles; which is ‡ absurd. Wherefore, a Right Line drawn from the Point E perpendicular to the Plane AB, does not fall without the Right Line AD; and so it must necessarily fall on it. Therefore, *if a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane; that Perpendicular shall fall in the common Section of the Planes*; which was to be demonstrated.

* *Def.* 4 *of this.*

† *Def.* 3. *of this.*

‡ 17. 1.

PRO-

PROPOSITION XXXIX.

THEOREM.

If the Sides of the opposite Planes of a solid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of those Planes, and the Diameter of the solid Parallelepipedon, shall divide each other into two equal Parts.

LET the Sides of CF, AH, the opposite Planes of the solid Parallelepipedon AF, be cut in Half in the Points K, L, M, N, X, O, P, R; and let the Planes KN, XR, be drawn thro' the Sections: Also, let YS be the common Section of the Planes, and DG the Diameter of the solid Parallelepipedon. I say, YS, DG, bisect each other; that is, YT is equal to TS, and DT to TG.

For, join DY, YE, BS, SG. Then, because DX is parallel to OE, the alternate Angles DXY, YOE,
* 29. are * equal to one another. And because DX is equal to OE, and YX to YO, and they contain equal An-
† 4. 1. gles, the Base DY shall be † equal to the Base YE, and the Triangle DXY to the Triangle YOE, and the other Angles equal to the other Angles: Therefore the Angle XYD is equal to the Angle OYE; and so
‡ 15. DYE is ‡ a Right Line. For the same Reason, BSG is also a Right Line, and BS is equal to SG; then, because CA is equal and parallel to DB, as also to EG, DB shall be equal and parallel to EG; and the Right
* 33. Lines DE, GB, join them: Therefore DE is * parallel to BG, and D, Y, G, S, are Points taken in each of them; and DG, YS, are joined: Therefore DG,
† 7 of this. YS, are † in one Plane. And since DE is parallel to
* 29. 1. BG, the Angle EDT shall be * equal to the Angle BGT, for they are alternate: But the Angle DTY is
‡ 15. 1. ‡ equal to the Angle GTS; therefore DTY, GTS, are two Triangles, having two Angles of the one equal to two Angles of the other, as likewise one Side of the one equal to one Side of the other; *viz.* the Side DY equal to the Side GS; for they are Halves of DE, BG;
* 26. 1. therefore * they shall have the other Sides of the one equal

to the other Sides of the other; and so DT is equal to [illegible]G, and Y[illegible] to TS. Wherefore, *if the Sides of the opposite Planes of a solid Parallelepipedon, be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of those Planes, and the Diameter of the solid Parallelepipedon, shall divide each other into two equal Parts*; which was to be demonstrated.

PROPOSITION XL.

Theorem.

Of two triangular Prisms, one standing on a Base which is a Parallelogram, and the other on a Triangle, if their Altitudes from these Bases are equal, and the Parallelogram double to the Triangle, then those Prisms are equal to each other.

LET ABCDEF, GHKLMN, be two Prisms of equal Altitude, the Base of one of which is the Parallelogram AF, and that of the other the Triangle GHK; and let the Parallelogram AF be double to the Triangle GHK. I say, the Prism ABCDEF is equal to the Prism GHKLMN.

For, compleat the Solids AX, GO. Then, because the Parallelogram AF is double to the Triangle GHK; and since the Parallelogram HK is * double to the Triangle GHK; the Parallelogram AF shall be equal to the Parallelogram HK. But solid Parallelepipedons, that stand upon equal Bases, and have the same Altitude, are † equal to one another; therefore the Solid AX is equal to the Solid GO. But the Prism ABCDEF is ‡ half the Solid AX; and the Prism GHKLMN is half the Solid GO; therefore the Prism ABCDEF is equal to the Prism GHKLMN. Wherefore, *if there be two triangular Prisms having equal Altitudes, the Base of one of which is a Parallelogram, and that of the other a Triangle; and if the Parallelogram be double to the Triangle, the said Prisms shall be equal to each other*; which was to be demonstrated.

* 41. 1.
† 31 *of this.*
‡ 28 *of this.*

The End *of the* Eleventh Book.

EUCLID's

EUCLID's ELEMENTS.

BOOK XII.

PROPOSITION I.

THEOREM.

Similar Polygons, inscribed in Circles, are to one another as the Squares of the Diameters of the Circles.

LET ABCDE, FGHKL, be Circles, wherein are inscribed the similar Polygons ABCDE, FGHKL; and let BM, GN, be Diameters of the Circles. I say, as the Square of BM is to the Square of GN, so is the Polygon ABCDE to the Polygon FGHKL.

For, join BE, AM, GL, FN. Then, because the Polygon ABCDE is similar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL: Therefore the two Triangles BAE, GFL, have one Angle of the one equal to one Angle of the other; *viz.* the Angle BAE equal to the Angle GFL, and the Sides about the equal Angles proportional. Wherefore the Triangle ABE is * equiangular to the Triangle FGL; and so the Angle AEB is equal to the Angle FLG: But the Angle AEB is † equal to the Angle AMB, for they stand on the same

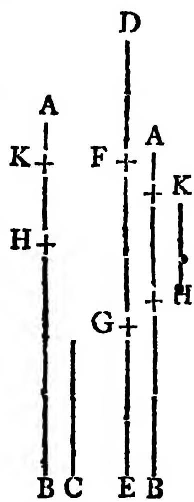
* 6. 6.

† 21. 3.

same Circumference; and the Angle FLG is † equal † 21. 3.
to the Angle FNG: Therefore the Angle AMB is equal to the Angle FNG. But the Right Angle BAM is ‡ equal to the Right Angle GFN; wherefore the ‡ 31. 3.
other Angle shall be equal to the other Angle: And so the Triangle AMB is equiangular to the Triangle FGN; and consequently * as BM is to GN, so is BA to GF. But † the Proportion of the Square of BM to the Square of GN, is duplicate of the Proportion of BM to GN, and the Proportion of the Polygon ABCDE to the Polygon FGHKL, is † duplicate of the † 20. 6.
Proportion of BA to GF: Wherefore, as the Square of BM is to the Square of GN, so is the Polygon ABCDE to the Polygon FGHKL. Therefore, *similar Polygons, inscribed in Circles, are to one another as the Squares of the Diameters of the Circles*; which was to be demonstrated.

LEMMA.

If there be two unequal Magnitudes proposed, and from the greater be taken a Part greater than its Half; and if from what remains there be again taken a Part greater than half this Remainder; and again from this last Remainder a Part greater than its Half; and if this be done continually; there will remain at last a Magnitude that shall be less than the lesser of the proposed Magnitudes.



LET AB and C be two unequal Magnitudes, whereof AB is the greater: I say, if from AB be taken a greater Part than Half, and from the Part remaining there be again taken a Part greater than its Half,

Half, and this be done continually, there will remain a Magnitude, at last, that shall be less than the Magnitude C.

For C being some Number of Times multiplied, will become greater than the Magnitude AB. Let it be multiplied, and let DE be a Multiple of C greater than AB; divide DE into Parts DF, FG, GE, each equal to C, and take BH, a Part greater than half of AB, from AB, and again from AH, the Part HK greater than half AH, and from AK a Part greater than half AK, and so on, until the Divisions that are in AB, are equal in Number to the Divisions in DE: Therefore let the Divisions AK, KH, HB, be equal in Number to the Divisions DF, FG, GE. Then, because DE is greater than AB, and the Part EG taken from ED is less than half thereof, and the Part BH, greater than half of AB, is taken from it; the Part remaining, DG, shall be greater than the Part remaining, HA. Again, because GD is greater than HA; and GF, being half of GD, is taken from the same; and HK, being greater than half HA, is taken from this likewise; the Part remaining, FD, shall be greater than the Part remaining, AK. But FD is equal to C; therefore C is greater than AK; and so the Magnitude AK is lesser than C: Therefore, *the Magnitude* AK *being the Part remaining of the Magnitude* AB, *is less than the lesser proposed Magnitude* C; which was to be demonstrated. If the Halves of the Magnitudes should have been taken, we demonstrate this after the same manner. *This is the first Proposition of the tenth Book.*

PROPOSITION II.

Theorem.

Circles are to each other as the Squares of their Diameters.

LET ABCD, EFGH, be Circles, whose Diameters are BD, FH. I say, as the Square of BD is to the Square of FH, so the Circle ABCD is to the Circle EFGH.

For,

For, if it be not so, the Square of BD shall be to the Square of FH, as the Circle ABCD is to some Space either less or greater than the Circle EFGH. First, let it be to a Space S, less than the Circle EFGH; and let the Square EFGH be described in the Circle.

This Square EFGH will be greater than half the Circle EFGH; because, if we draw Tangents to the Circle thro' the Points E, F, G, H, the Square EFGH will be half that described about the Circle: But the Circle is less than the Square described about it; therefore the Square EFGH is greater than half the Circle EFGH. Let the Circumferences EF, FG, GH, HE, be bisected in the Points K, L, M, N; and join EK, KF, FL, LG, GM, MH, HN, NE: Then each of the Triangles EKF, FLG, GMH, HNE, will be * greater than one half of the Segment of the Circle it stands in; because, if Tangents at the Circle be drawn thro' the Points K, L, M, N, and the Parallelograms that are on the Right Lines DF, FG, GH, HE, be compleated, each of the Triangles EKF, FLG, GMH, HNE, is half of each of the corresponding Parallelograms: But the Segment is less than the Parallelogram; wherefore each of the Triangles EKF, FLG, GMH, HNE, is greater than one half of the Segment of the Circle in which it stands: Therefore, if these Circumferences be again bisected, and Right Lines be drawn joining the Points of Bisection, and you do thus continually, there will at last remain Segments of the Circle, that shall be less than the Excess, by which the Circle EFGH exceeds the Space S. For it is demonstrated, in the foregoing *Lemma*, that, if there be two unequal Magnitudes proposed, and if from the greater a Part greater than half be taken, and again from the Part remaining a Part greater than half be taken, and you do thus continually, there will at last remain a Magnitude that will be less than the lesser proposed Magnitude. Let the Segments of the Circle EFGH, on the Right Lines EK, KF, FL, LG, GM, MH, HN, NE, be those which are less than the Excess, whereby the Circle EFGH exceeds the Space S; and then the remaining Polygon EKFLGMHN shall be greater than the Space S. Also, describe the Polygon AXBOCPDR in the Circle ABCD, similar to the

* 41. 1.

R Polygon

Polygon EKFLGMHN. Wherefore, as the Square of BD is to the Square of FH, so is the Polygon
* 1 of this. AXBOCPDR to the * Polygon EKFLGMHN. But as the Square of BD is to the Square of FH, so is the
‡ From Hyp Circle ABCD to the Space S ‡: Wherefore, as the
† 11. 5. Circle ABCD is to the Space S, so is † the Polygon AXBOCPDR to the Polygon EKFLGMHN. But the Circle ABCD is greater than the Polygon in it;
* 14. 5. wherefore the Space S shall be * also greater than the
‡ From Hyp. Polygon EKFLGMHN: But it is less ‡ likewise; which is absurd; therefore the Square of BD to the Square of FH, is not as the Circle ABCD to some Space less than the Circle EFGH. After the same manner we likewise demonstrate, that the Square of FH to the Square of BD, is not as the Circle EFGH to some Space T less than the Circle ABCD. Lastly, I say, the Square of BD to the Square of FG, is not as the Circle ABCD to some Space greater than the Circle EFGH: For, if it be possible, let it be so, and let the Space S be greater than the Circle EFGH: Then it shall be (by Inversion), as the Square of FG is to the Square of BD, so is the Space S to the Circle ABCD. But, because S is greater than the Circle EFGH, the Space S shall be to the Circle ABCD, as the Circle EFGH is to some Space T less than the Circle ABCD. Therefore, as
11. 5. the Square of FH is to the Square of BD, so is * the Circle EFGH to some Space T less than the Circle ABCD, which has been demonstrated to be impossible; wherefore the Square of BD to the Square of FH, is not as the Circle ABCD to some Space greater than the Circle EFGH: But it also has been proved, that the Square of BD to the Square of FH, is not as the Circle ABCD to some Space less than the Circle EFGH: Wherefore, as the Square of BD is to the Square of FH, so shall the Circle ABCD be to the Circle EFGH. Wherefore, *Circles are to each other as the Squares of their Diameters*; which was to be demonstrated.

PRO-

PROPOSITION III.

Theorem.

Every Pyramid, having a triangular Base, may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid; and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid.

LET there be a Pyramid, whose Base is the Triangle ABC, and Vertex the Point D. I say, the Pyramid ABCD may be divided into two Pyramids equal and similar to one another, having triangular Bases, and similar to the Whole; and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid.

For, bisect AB, BC, CA, AD, DB, DC, in the Points E, F, G, H, K, L; and join EH, EG, GH, HK, KL, LH, EK, KF, FG: Then, because AE is equal to EB, and AH to HD; EH shall be * parallel to DB; for the same Reason, HK also is parallel to AB; therefore HEBK is a Parallelogram; and so HK is equal † to EB: But EB is equal to AE; therefore AE shall be also equal to HK; but AH is equal to HD; wherefore the two Sides AE, AH, are equal to the two Sides KH, HD, each to each, and the Angle EAH is ‡ equal to the Angle KHD; wherefore the Base EH is * equal to the Base KD; and so the Triangle AEH is equal and similar to the Triangle HKD. For the same Reason, the Triangle AHG shall also be equal and similar to the Triangle HDL; and because the two Right Lines EH, HG, touching each other, are parallel to the two Right Lines KD, DL, touching each other, and not in the same Plane with them, they shall contain † equal Angles. Therefore the Angle EHG is equal to the Angle KDL. Again, because the two Sides EH, HG, are equal to the two Sides KD, DL, each to each; and the Angle EHG is equal to the Angle KDL; the Base EG shall be * equal to the Base KL; and therefore the Triangle EHG is equal and similar to the Triangle KDL. For the same Reason, the Triangle

* 2. 6.

† 34. 1.

‡ 29. 1.

* 4. 1.

† 10. 11

* 4. 1.

angle AEG is also equal and similar to the Triangle HKL; wherefore the Pyramid whose Base is the Triangle AEG, and Vertex the Point H, ‡ is equal and similar to the Pyramid whose Base is the Triangle HKL, and Vertex the Point D. And because HK is drawn parallel to the Side AB of the Triangle ADB, the Triangle ADB * shall be equiangular to the Triangle DKH, and they have their Sides proportional; therefore the Triangle ADB is similar to the Triangle DHK. And, for the same Reason, the Triangle DBC is similar to the Triangle DKL; and the Triangle ADC to the Triangle DHL. And since the two Right Lines BA, AC, touching each other, are parallel to the two Lines KH, HL, touching each other, not being in the same Plane with them, these shall contain equal Angles †; therefore the Angle BAC is equal to the Angle KHL: And BA is to AC, as KA is to HL; wherefore the Triangle ABC is similar to the Triangle HKL; and so the Pyramid whose Base is the Triangle ABC, and Vertex the Point D, is similar to the Pyramid, whose Base is the Triangle HKL, and Vertex the Point D. But the Pyramid whose Base is the Triangle HKL, and Vertex the Point D, has been proved similar to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H; therefore the Pyramid whose Base is the Triangle ABC, and Vertex the Point D, is similar to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H: Wherefore both the Pyramids AEGH, HKLD, are similar to the whole Pyramid ABCD. And because BF is equal to FC, the Parallelogram EBFG will be double to the Triangle GFC; and since there are two Prisms of equal Altitude, one of which has that Parallelogram for a Base, and the other the Triangle, and the Parallelogram is double to the Triangle; those Prisms will be * equal to one another: Therefore the Prism contained under the two Triangles BKF, EHG, and the three Parallelograms EBFG, EBKH, KHGF, is equal to the Prisms contained under the two Triangles GFC, HKL, and the three Parallelograms KFCL, LCGH, HKFG: And it is manifest, that each of those Prisms, the Base of one of which is the Parallelogram EBGF, and the opposite Base to that the Right Line KH, and the Base of the other, the Triangle GFC, and the opposite

Base

‡ Def. 10. 11.

† 10. 11.

* 40. 11.

Base to this the Triangle KLH, are greater than either of the Pyramids, whose Bases are the Triangles AEG, HKL, and Vertices the Points H and D. For since, if the Right Lines EF, EH, be joined, the Prism, whose Base is the Parallelogram EBFG, and the opposite Base to that the Right Line KH, is greater than the Pyramid, whose Base is the Triangle EBF, and Vertex the Point K. But the Pyramid whose Base is the Triangle EBF, and Vertex the Point K, is equal to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H; for they are contained under equal and similar Planes. Wherefore the Prism whose Base is the Parallelogram EBFG, and the opposite Base to it the Right Line HK, is greater than the Pyramid whose Base is the Triangle AEG, and Vertex the Point H. But the Prism, whose Base is the Parallelogram EBFG, and the opposite Base to it the Right Line HK, is equal to the Prism whose Base is the Triangle GFC, and the opposite Base to this the Triangle HKL; and the Pyramid, whose Base is the Triangle AEG, and Vertex the Point H, is equal to the Pyramid whose Base is the Triangle HKL, and Vertex the Point D: Therefore the two Prisms aforesaid are greater than the said two Pyramids, whose Bases are the Triangles AEG, HKL, and Vertices the Points H, D: And so the whole Pyramid, whose Base is the Triangle ABC, and Vertex the Point D, is divided into two equal Pyramids, similar to each other, and to the Whole, and into two equal Prisms, which two Prisms, together, are greater than half of the whole Pyramid. Therefore, *every Pyramid, having a triangular Base, may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid; and into two equal Prisms, which two Prisms are greater than the Half of the whole Pyramid*; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If there are two Pyramids of the same Altitude, having triangular Bases, and each of them be divided into two Pyramids, equal to one another, and similar to the Whole, as also into two equal Prisms; and if, in like manner, each of the two Pyramids, made by the former Division, be divided, and this be done continually; then, as the Base of one Pyramid is to the Base of the other Pyramid, so are all the Prisms that are in one Pyramid, to all the Prisms that are in the other Pyramid, being equal in Multitude.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC, DEF, whose Vertices are the Points G, H; and let each of them be divided into two Pyramids, equal to one another, and similar to the Whole, and into two equal Prisms; and if, in like manner, each of the Pyramids, made by the former Division, be conceived to be divided, and this be done continually; I say, as the Base ABC is to the Base DEF, so are all the Prisms that are in the Pyramid ABCG, to all the Prisms that are in the Pyramid DEFH, being equal in Multitude.

For, since BX is equal to XC, and AL to LC, XL shall be * parallel to AB, and the Triangle ABC similar to the Triangle LXC. For the same Reason, the Triangle DEF shall be also similar to the Triangle RQF. And because BC is double to CX, and EF to FQ, it shall be, as BC is to CX, so is EF to FQ. And since there are described upon BC, CX, Right-lined Figures ABC, LXC, similar and alike situate; and upon EF, FQ, Right-lined Figures DEF, RQF, similar and alike situate; therefore, as the Triangle BAC is to the Triangle LXC, so is † the Triangle DEF to the Triangle RQF, and (by Alternation) as the Triangle ABC is to the Triangle DEF, so is the Triangle LXC to the Triangle RQF. But as the Triangle LXC is to the Triangle RQF, so is ‡ the Prism, whose Base is the Triangle LXC, and the opposite Base

* 2. 6.

† 22. 6.

‡ 28. and 32. 11.

to

to that the Triangle OMN, to the Prism, whose Base is the Triangle RQF, and the opposite Base to that the Triangle STY; therefore, as the Triangle ABC is to the Triangle DEF, so is * the Prism whose Base is the Triangle LXC, and the opposite Base to that the Triangle OMN, to the Prism whose Base is the Triangle RQF, and the opposite Base to that the Triangle STY: And because the two Prisms that are in the Pyramid ABCG are equal to one another, as also those two that are in the Pyramid DEFH; it shall be, as the Prism whose Base is the Parallelogram KLXB, and the opposite Base to that the Right Line MO, is to the Prism whose Base is the Triangle LXC, and the opposite Base to that the Triangle OMN, so is the Prism whose Base is the Parallelogram EPRQ, and the opposite Base to that the Right Line ST, to the Prism whose Base is the Triangle RQF, and the opposite Base to that the Triangle STY: Therefore (by compounding), as the Prisms KBXLMO, LXCMNO, together, are to the Prism LXCMNO, so the Prisms PEQRST, RQFSTY, together, are to the Prism RQFSTY: And (by Alternation), as the Prisms KBXLMO, LXCMNO, together, are to the Prisms PEQRST, RQFSTY, together, so is the Prism LXCMNO to the Prism RQFSTY: But as the Prism LXCMNO is to the Prism RQFSTY, so has the Base LXC been proved to be to the Base RFQ; and so the Base ABC to the Base DEF: Therefore, also, as the Triangle ABC is to the Triangle DEF, so are the two Prisms that are in the Pyramid ABCG, to the two Prisms that are in the Pyramid DEFH. If, in the same manner, each of the Pyramids OMNG, STYH, made by the former Division, be divided, it shall be, as the Base OMN is to the Base STY, so the two Prisms that are in the Pyramid OMNG, to the two Prisms that are in the Pyramid STYH. But as the Base OMN is to the Base STY, so is the Base ABC to the Base DEF: Therefore, as the Base ABC is to the Base DEF, so are the two Prisms that are in the Pyramid ABCG, to the two Prisms that are in the Pyramid DEFH; and so the two Prisms that are in the Pyramid OMNG, to the two Prisms that are in the Pyramid STYH; and so the four to the four. We demonstrate the same of Prisms made by the Division

* 11. 5.

of the Pyramids AKLO, DPRS; and of the Prisms, being equal in Multitude; which was to be demonstrated.

PROPOSITION V.

Theorem.

Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC, DEF, whose Vertices are the Points G, H. I say, as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Pyramid DEFH.

For, if it be not so, then it shall be as the Base ABC is to the Base DEF, so is the Pyramid ABCG to some Solid, greater or less than the Pyramid DEFH. First, let it be to a Solid less, which let be Z; and divide the Pyramid DEFH into two Pyramids equal to each other, and similar to the Whole, and into two equal Prisms; then these two Prisms are greater than the Half of the whole Pyramid: And, again, let the Pyramids, made by the former Division, be divided after the same manner; and let this be done continually, until the Pyramids in the Pyramid DEFH are less than the Excess by which the Pyramid DEFH, exceeds the Solid Z. Let these, for Example, be the Pyramids DPRS, STYH; then the Prisms remaining in the Pyramid DEFH, are greater than the Solid Z: Also, let the Pyramid ABCG be divided into the same Number of similar Parts as the Pyramid DEFH is; and then, * 4 of this. as the Base ABC is to the Base DEF, so * the Prisms that are in the Pyramid ABCG, to the Prisms that are in the Pyramid DEFH. But as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Solid ‡ By Hyp. Z ‡; and therefore, as the Pyramid ABCG is to the Solid Z, so are the Prisms that are in the Pyramid ABCG, to the Prisms that are in the Pyramid DEFH. But the Pyramid ABCG is greater than the Prisms that are in it; wherefore, also, the Solid Z is greater than the

ABC, than that are in the Pyramid DEFH. But it is less also; which is absurd; therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH. After the same manner we demonstrate, that the Base DEF to the Base ABC, is not as the Pyramid DEFH to some Solid less than the Pyramid ABCG: Therefore, I say, neither is the Base ABC to the Base DEF, as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. For, if this be possible, let it be to the Solid I, greater than the Pyramid DEFH; then (by Inversion) the Base DEF shall be to the Base ABC, as the Solid I to the Pyramid ABCG: But since the Solid I is greater than the Pyramid DEFH, it shall be, as the Solid I is to the Pyramid ABCG, so is the Pyramid DEFH to some Solid less than the Pyramid ABCG; and so, as the Base DEF is to the Base ABC, so is the Pyramid DEFH to some Solid less than the Pyramid ABCG, which is absurd, as just now has been proved: Therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. But it has been also proved, that the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH; wherefore, as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Pyramid DEFH. Therefore, *Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases*; which was to be demonstrated.

From what has been befor proved.

PROPOSITION VI.

Theorem.

Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases.

LET there be Pyramids of the same Altitude, which have the polygonous Bases ABCDE, FGHKL; and let their Vertices be the Points M, N. I say, as the Base ABCDE is to the Base FGHKL, so is the Pyramid ABCDEM to the Pyramid FGHKLN.

For,

For, let the Base ABCDE be divided into Triangles ABC, ACD, ADE; and the Base FGHKL into the Triangles FGH, FHK, FKL; and let Pyramids be conceived upon every one of those Triangles, of the same Altitude with the Pyramids ABCDEM, FGHKLN: Then, because the Triangle ABC is to the Triangle ACD, as * the Pyramid ABCM is to the Pyramid ACDM; and (by compounding) as the Trapezium ABCD is to the Triangle ACD, so is the Pyramid ABCDM to the Pyramid ACDM: But as the Triangle ACD is to the Triangle ADE, so is * the Pyramid ACDM to the Pyramid ADEM. Wherefore (by Equality of Proportion), as the Base ABCD is to the Base ADE, so is the Pyramid ABCDM to the Pyramid ADEM: And again (by Composition of Proportion), as the Base ABCDE is to the Base ADE, so is the Pyramid ABCDEM to the Pyramid ADEM. For the same Reason, as the Base FGHKL is to the Base FKL, so is the Pyramid FGHKLN to the Pyramid FKLN: And since there are two Pyramids ADEN, FKLN, having triangular Bases, and the same Altitude; the Base ADE shall be * to the Base FKL, as the Pyramid ADEM to the Pyramid FKLN: And since the Base ABCDE is to the Base ADE, as the Pyramid ABCDEM is to the Pyramid ADEM; and as the Base ADE is to the Base FKL, so is the Pyramid ADEM to the Pyramid FKLN; it shall be (by Equality of Proportion), as the Base ABCDE is to the Base FKL, so is the Pyramid ABCDEM to the Pyramid FKLN: But as the Base FKL is to the Base FGHKL, so was the Pyramid FKLN to the Pyramid FGHKLN. Wherefore, again, (by Equality of Proportion), as the Base ABCDE is to the Base FGHKL, so is the Pyramid ABCDEM to the Pyramid FGHKLN. Therefore, *Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases*; which was to be demonstrated.

* 5 of this.

If the Bases had not consisted of equal Numbers of Sides, the Demonstration had been the same.

PRO-

PROPOSITION VII.

THEOREM.

Every Prism, having a triangular Base, may be divided into three Pyramids, equal to one another, and having triangular Bases.

LET there be a Prism whose Base is the Triangle ABC, and the opposite Base to that the Triangle DEF. I say, the Prism ABCDEF may be divided into three equal Pyramids, that have triangular Bases.

For, join BD, EC, CD: Then, because ABED is a Parallelogram, whose Diameter is BD, the Triangle ABD shall be* equal to the Triangle EBD. Therefore the Pyramid whose Base is the Triangle ABD, and Vertex the Point C, is † equal to the Pyramid whose Base is the Triangle EDB, and Vertex the Point C. But the Pyramid, whose Base is the Triangle EDB, and Vertex the Point C, is the same as the Pyramid whose Base is the Triangle EBC, and Vertex the Point D; for they are contained under the same Planes: Therefore the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, is equal to the Pyramid whose Base is the Triangle EBC, and Vertex the Point D. Again, because FCBE is a Parallelogram, whose Diameter is CE, the Triangle ECF shall be* equal to the Triangle CBE; and so the Pyramid whose Base is the Triangle BEC, and Vertex the Point D, is † equal to the Pyramid whose Base is the Triangle ECF, and Vertex the Point D. But the Pyramid, whose Base is the Triangle BCE, and Vertex the Point D, has been proved equal to the Pyramid whose Base is the Triangle ABD, and Vertex the Point C: Wherefore, also, the Pyramid, whose Base is the Triangle CEF, and Vertex the Point D, is equal to the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C: Therefore, *the Prism* ABCDEF, *is divided into three Pyramids equal to one another, and having triangular Bases.* And because the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, is the same with the Pyramid whose Base is the Triangle CAB, and Vertex the Point D; for they

* 34. 1.
† 6 of this.

they are contained under the same Planes; and the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, has been proved to be a third Part of the Prism, whose Base is the Triangle ABC, and the opposite Base to that the Triangle DEF: Therefore, also, *the Pyramid, whose Base is the Triangle* ABC, *and Vertex the Point* D, *is a third Part of the Prism, having the same Base,* viz. *the Triangle* ABC, *and the opposite Base the Triangle* DEF; which was to be demonstrated.

Coroll. 1. It is manifest from hence, that every Pyramid is a third Part of a Prism, having the same Base, and an equal Altitude; because, if the Base of a Prism, as also the opposite Base, be of any other Figure, it may be divided into Prisms having triangular Bases.

2. Prisms of the same Altitude are to one another as their Bases.

PROPOSITION VIII.

THEOREM.

Similar Pyramids, having triangular Bases, are in a triplicate Proportion of their homologous Sides.

LET there be two Pyramids similar and alike situate, having the triangular Bases ABC, DEF; and let their Vertices be the Points G, A. I say the Pyramid ABCG to the Pyramid DEFH, has a Proportion triplicate of that which BC has to EF.

For, compleat the solid Parallelepipedon BGML, EHPO; then, because the Pyramid ABCG is simi-
* Def. 9. 11. lar to the Pyramid DEFA, the Angle ABC shall be * equal to the Angle DEF, the Angle GBC equal to the Angle HEF, and the Angle ABG equal to the Angle DEH: And AB is to DE, as BC is to EF; and so is BG to EH. Therefore because the Angle ABC is equal to the Angle DEF; and the Sides about the equal Angles are proportional; the Parallelogram BM
† 6. 6. shall be † similar to the Parallelogram EP. For the same Reason, the Parallelogram BN is similar to the

Paral-

rallelogram ER, and the Parallelogram BK to the Parallelogram EX. Therefore three Parallelograms BM, BK, BN, are ſimilar to three Parallelograms EP, EX, ER. But the three Parallelograms BM, BK, BN, are equal and ſimilar to the three oppoſite ones; as alſo the three Parallelograms EP, EX, ER: Therefore the Solids BGML, EHPO, are contained under equal Numbers of ſimilar and equal Planes; and, conſequently, the Solid BGML is ſimilar to the Solid EHPO. But ſimilar ſolid Parallelepipedons are * to each other in a triplicate Proportion of their homologous Sides; therefore the Solid BGML to the Solid EHPO, has a Proportion triplicate of that which the homologous Side BC has to the homologous Side EF. But as the Solid BGML is to the Solid EHPO, ſo is † the Pyramid ABCG to the Pyramid DEFH; for the Pyramid is the one ſixth Part of that Solid, ſince the Priſm, which is the Half of the ſolid Parallelepipedon, is triple of the Pyramid. Wherefore, *the Pyramid,* ABCG *to the Pyramid* DEFH, *ſhall have a triplicate Proportion to that which* BC *has to* EF; which was to be demonſtrated.

* 33. 11.

† 15. 5.

Coroll. From hence it is manifeſt, that ſimilar Pyramids having polygonous Baſes, are to one another in a triplicate Proportion of their homologous Sides. For, if they be divided into Pyramids having triangular Baſes; becauſe their ſimilar polygonous Baſes are divided into ſimilar Triangles equal in Number, and homologous to the Wholes; it ſhall be, as one Pyramid, having a triangular Baſe in one of the Pyramids, is to a Pyramid having a triangular Baſe in the other Pyramid; ſo are all the Pyramids, having triangular Baſes in one Pyramid, to all the Pyramids having triangular Baſes in the other Pyramid; that is, ſo is one of the Pyramids, having the polygonous Baſe, to the other: But a Pyramid, having a triangular Baſe, to a Pyramid having a triangular Baſe, is in a triplicate Proportion of the homologous Sides. Therefore one Pyramid, having a polygonous Baſe, to another Pyramid having a ſimilar Baſe, is in a triplicate Proportion of their homologous Sides.

PROPOSITION IX.

THEOREM.

The Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal.

LET there be equal Pyramids, having the triangular Bases ABC, DEF, and Vertices the Points G, H. I say, the Bases and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional; that is, as the Base ABC is to the Base DEF, so is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG.

For, compleat the solid Parallelepipedons BGML, EHPO; then, because the Pyramid ABCG is equal to the Pyramid DEFH; and the Solid BGML is sextuple the Pyramid ABCG; and the Solid EHPO sextuple the Pyramid DEFH; the Solid BGML shall be * equal to the Solid EHPO. But the Bases and Altitudes of equal solid Parallelepipedons are reciprocally proportional; therefore, as the Base BM is to the
† 34. 11. Base EP, so is † the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But as the Base BM
* 15. 5. is to the Base EP, so is * the Triangle ABC to the Triangle DEF; therefore, as the Triangle ABC is to the Triangle DEF, so is the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But the Altitude of the Solid EHPO is the same as the Altitude of the Pyramid DEFH; and the Altitude of the Solid BGML, the same as the Altitude of the Pyramid ABCG; therefore, as the Base ABC is to the Base DEF, so is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG: Wherefore the Bases and Altitudes of the equal Pyramids ABCG, DEFH, are reciprocally proportional. And if the Bases and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional; that is, if the Base ABC to the Base DEF, be as the Altitude of the Pyramid DEFH

E F e C ... the Altitude of the Pyramid ABCG; I say, ... Pyramid ABCG is equal to the Pyramid DEFH: For, the same Construction remaining, because the Base ABC to the Base DEF, is as the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG; and as the Base ABC is to the Base DEF, so is the Parallelogram BM to the Parallelogram EP; therefore the Parallelogram BM to the Parallelogram EP, shall be also as the Altitude of the Pyramid DEFH is to the Altitude of the Pyramid ABCG. But as the Altitude of the Pyramid DEFH is the same as the Altitude of the solid Parallelepipedon EHPO, and the Altitude of the Pyramid ABCG, the same as the Altitude of the solid Parallelepipedon BGML; therefore the Base BM to the Base EP, will be as the Altitude of the solid Parallelepipedon EHPO to the Altitude of the solid Parallelepipedon BGML. But those solid Parallelepipedons, whose Bases and Altitudes are reciprocally proportional, are † equal to each other; therefore the solid Parallelepipedon BGML, is equal to the solid Parallelepipedon EHPO: Now the Pyramid ABCG is a sixth Part of the Solid BGML; and, in like manner, the Pyramid DEFH is a sixth Part of the Solid EHPO; therefore the Pyramid ABCG is equal to the Pyramid DEFH. Wherefore, *the Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal*; which was to be demonstrated. † 34. 11.

PROPOSITION X.

Theorem.

Every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude.

LET a Cone have the same Base as a Cylinder; *viz.* the Circle ABCD, and an Altitude equal to it. I say, the Cone is a third Part of the Cylinder; that is, the Cylinder is triple to the Cone.

For,

For, if the Cylinder be not triple to [illegible] [illegible]e, shall be greater or less than triple thereof. [illegible] be greater than triple to the Cone, and let the Square ABCD be described in the Circle ABCD; then the Square ABCD, is greater than one half of the Circle ABCD. Now let a Prism be erected upon the Square ABCD, having the same Altitude as the Cylinder, and this Prism will be greater than one half of the Cylinder; because, if a Square be circumscribed about the Circle ABCD, the inscribed Square will be one half of the circumscribed Square; and if a Prism be erected upon the circumscribed Square of the same Altitude of the same Cylinder, since Prisms are to * one another as their Bases, the Prism erected upon the Square ABCD, is one half of the Prism erected upon the Square described about the Circle ABCD. But the Cylinder is less than the Prism erected on the Square described about the Circle ABCD; therefore the Prism erected on the Square ABCD, having the same Height as the Cylinder, is greater than one half of the Cylinder. Let the Circumferences AB, BC, CD, DA, be bisected in the Points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: Then each of the Triangles AEB, BFC, CGD, DHA, is † greater than the half of each of the Segments in which they stand. Let Prisms be erected from each of the Triangles AEB, BFC, CGD, DHA, of the same Altitude as the Cylinder; then every one of these Prisms erected is greater than half its correspondent Segment of the Cylinder. For, because, if Parallels be drawn thro' the Points E, F, G, H, to AB, BC, CD, DA, and Parallelograms be compleated on the said AB, BC, CD, DA, on which are erected solid Parallelepipedons of the same Altitude as the Cylinder; then each of those Prisms that are on the Triangles AEB, BFC, CGD, DHA, are Halves ‡ of each of the solid Parallelepipedons; and the Segments of the Cylinder are less than the erected solid Parallelepipedons; and, consequently, the Prisms that are on the Triangles AEB, BFC, CGD, DHA, are greater than the Halves of the Segments of the Cylinder: And so, bisecting the other Circumferences, joining Right Lines, and on every one of the Triangles erecting Prisms of the same Height as the

* 2 Cor. 7. of this.

† This follows from 2 of this.

‡ 32. 11.

the Cylinder, and doing this continually, we ſhall at laſt leave certain Portions of the Cylinder left, that are leſs than the Exceſs by which the Cylinder exceeds triple the Cone.

Now, let theſe Portions remaining be AE, EB, BF, FC, CG, GD, DH, HA; then the Priſm remaining, whoſe Baſe is the Polygon AEBFCGDH, and Altitude equal to that of the Cylinder's is greater than the Triple of the Cone. But the Priſm, whoſe Baſe is the Polygon AEBFCGDH, and Altitude the ſame as that of the Cylinder's, is * triple of the Pyramid, whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone; and therefore the Pyramid, whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone, is greater than the Cone, whoſe Baſe is the Circle ABCD: But it is leſs alſo (for it is comprehended by it) which is abſurd; therefore the Cylinder is not greater than triple the Cone. I ſay, it is neither leſs than triple the Cone: For, if it be poſſible, let the Cylinder be leſs than triple the Cone; than (by Inverſion) the Cone ſhall be greater than a third Part of the Cylinder: Let the Square ABCD be deſcribed in the Circle ABCD; then the Square ABCD is greater than half of the Circle ABCD: And let a Pyramid be erected on the Square ABCD, having the ſame Vertex as the Cone; then the Pyramid erected is greater than one half of the Cone; becauſe, as has been already demonſtrated, if a Square be deſcribed about the Circle, the Square ABCD ſhall be half thereof: And if ſolid Parallelepipedons be erected upon the Squares of the ſame Altitude as the Cone, which are alſo Priſms; then the Priſm erected on the Square ABCD is one half of that erected on the Square deſcribed about the Circle; for they are to each other as their Baſes, and ſo likewiſe are their third Parts: Therefore the Pyramid, whoſe Baſe is the Square ABCD, is one half of that Pyramid erected upon the Square deſcribed about the Circle. But the Pyramid erected upon the Square deſcribed about the Circle is greater than the Cone, for it comprehends it; therefore the Pyramid, whoſe Baſe is the Square ABCD, and Vertex the ſame as that of the Cone, is greater than one half of the Cone.

* 1 Cor. 7. of this.

Bisect the Circumferences AB, BC, CD, DA, in the Points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA; and then each of the Triangles AEB, BFC, CGD, DHA, is greater than one half of each of the Segments they are in. Let Pyramids be erected upon each of the Triangles AEB, BFC, CGD, DHA, having the same Vertex as the Cone; then each of these Pyramids, thus erected, is greater than one half of the Segment of the Cone in which it is; and so, bisecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon every of the Triangles having the same Altitude as the Cone, and doing this continually, we shall at last have Segments of the Cone left, that will be less than the Excess by which the Cone exceeds the one third Part of the Cylinder: Let these Segments be those that are on AE, EB, BF, FC, CG, GD, DH, HA; and then the remaining Pyramid, whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is greater than a third Part of the Cylinder: But the Pyramid, whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is one third Part of the Prism whose Base is the Polygon AEBFCGDH, and Altitude the same as that of the Cylinder: Therefore the Prism, whose Base is the Polygon AEBFCGDH, and Altitude the same as that of the Cylinder, is greater than the Cylinder, whose Base is the Circle ABCD; but it is less also (as being comprehended thereby); which is absurd; therefore the Cylinder is not less than triple of the Cone: But it has been proved also not to be greater than triple of the Cone; therefore the Cylinder is necessarily triple of the Cone. Wherefore, *every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude*; which was to be demonstrated.

PRO-

PROPOSITION XI.

THEOREM.

Cones and Cylinders, of the same Altitude, are to one another as their Bases.

LET there be Cones and Cylinders of the same Altitude, whose Bases are the Circles ABCD, EFGH, Axes KL, MN, and Diameters of the Bases AC, EG. I say, as the Circle ABCD is to the Circle EFGH, so is the Cone AL to the Cone EN.

For, if it be not so, it shall be, as the Circle ABCD is to the Circle EFGH, so is the Cone AL to some Solid either less or greater than the Cone EN. First, let it be to the Solid X less than the Cone; and let the Solid I be equal to the Excess of the Cone EN above the Solid X: Then the Cone EN is equal to the Solids X and I. Let the Square EFGH be described in the Circle EFGH, which Square is greater than one half of the Circle, and erect a Pyramid upon the Square EFGH, of the same Altitude as the Cone; therefore the Pyramid erected is greater than one half of the Cone. For if we describe a Square about the Circle, and a Pyramid be erected thereon, of the same Altitude as the Cone; the Pyramid inscribed will be one half of the Pyramid circumscribed; for they are * to one another as their Bases; and their Cone is less than the circumscribed Pyramid: Therefore the Pyramid, whose Base is the Square EFGH, and Vertex the same as that of the Cone, is greater than one half of the Cone. Bisect the Circumferences EF, FG, GH, HE, in the Points P, R, S, O; and join HO, OE, EP, PF, FR, RG, GS, SH; then each of the Triangles HOE, EPF, FRG, GSH, is greater than one half of the Segment of the Circle wherein it is. Let a Pyramid be raised upon every one of the Triangles HOE, EPF, FRG, GSH, of the same Altitude as the Cone; then each of those erected Pyramids is greater than one half of its correspondent Segment of the Cone: And so bisecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon each of the Triangles, of the same Altitude

* 6 of this.

Altitude as that of the Cone; and doing this continually, there will at last be left Segments of the Cone that will together be less than the Solid I. Let those be the Segments that are on HO, OE, EP, PF, FR, RG, GS, SH; therefore the Pyramid remaining, whose Base is the Polygon HOEPFRGS, and Altitude the same as that of the Cone, is greater than the Solid X. Let the Polygon DTAYBQCV be described in the Circle ABCD, similar and alike situate to the Polygon HOEPFRGS; and let a Pyramid be erected thereon of the same Altitude as the Cone AL: Then, because the Square of AC to the Square of EG, is * as the Polygon DTAYBQCV to the Polygon HOEPFRGS; and the Square of AC is † to the Square of EG, as the Circle ABCD is to the Circle EFGH; it shall be, as the Circle ABCD is to the Circle EFGH, so is the Polygon DTAYBQCV to the Polygon HOEPFRGS. But as the Circle ABCD is to the Circle EFGH, so is the Cone AL to the Solid X (by *Hyp.*): And as the Polygon DTAYBQCV is to the Polygon HOEPFRGS, so is ‡ the Pyramid, whose Base is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whose Base is the Polygon HOEPFRGS, and Vertex the Point N. Therefore, as the Cone AL is to the Solid X, so is the Pyramid, whose Base is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whose Base is the Polygon HOEPFRGS, and Vertex the Point N. But the Cone AL is greater than the Pyramid that is in it; therefore the Solid X is greater than the Pyramid that is in the Cone EN; but it was put less, which is absurd. Therefore the Circle ABCD to the Circle EFGH, is not as the Cone AL to some Solid less than the Cone EN. In like manner it is demonstrated, that the Circle EFGH to the Circle ABCD, is not as the Cone EN to some Solid less than the Cone AL: I say, moreover, that the Circle ABCD to the Circle EFGH, is not as the Cone AL to some Solid greater than the Cone EN. For, if it be possible, let it be to the Solid Z greater than the Cone; then (by Inversion), as the Circle EFGH is to the Circle ABCD, so shall the Solid Z be to the Cone AL. But since the Solid Z is greater than the Cone EN, it shall be, as the Solid Z is to the Cone AL, so is the Cone

* 1 of this.
† 2 of this.
‡ 6 of this.

Cone EN to some Solid less than the Cone AL; and therefore, as the Circle EFGH is to the Circle ABCD, so is the Cone EN to some Solid less than the Cone AL; which has been proved to be impossible. Therefore the Solid ABCD to the Circle EFGH, is not as the Cone AL to some Solid greater than the Cone EN. It has also been proved, that the Circle ABCD to the Circle EFGH, is not as the Cone AL to some Solid less than the Cone EN; therefore, as the Circle ABCD is to the Circle EFGH, so is the Cone AL to the Cone EL: But as Cone is to Cone, so is * Cylinder to Cylinder; for each Cylinder is triple of * 15. 5.
each Cone; and therefore, as the Circle ABCD is to the Circle EFGH, so are Cylinders and Cones standing on them, of the same Altitude. Wherefore, *Cones and Cylinders of the same Altitude, are to one another as their Bases*; which was to be demonstrated.

PROPOSITION XII.

THEOREM.

Similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Bases.

LET there be similar Cones and Cylinders, whose Bases are the Circles ABCD, EFGH, and Diameters of the Bases BD, FH, and Axes of the Cones or Cylinders KL, MN. I say, the Cone, whose Base is the Circle ABCD, and Vertex the Point L, to the Cone whose Base is the Circle EFGH, and Vertex the Point N, hath a triplicate Proportion of that which BD has to FH.

For, if the Cone ABCDL to the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH; the Cone ABCDL shall have that triplicate Proportion to some Solid, either less or greater than the Cone EFGHN. First, let it have that triplicate Proportion to the Solid X, less than the Cone EFGHN; and let the Square EFGH be described in the Circle EFGH, which will be greater than one half of the Circle EFGH; and erect a Pyramid on the Square

EFGH, of the ſame Altitude with the Cone, then that Pyramid is greater than one half of the Cone. And ſo let the Circumferences EF, FG, GH, HE, be biſected in the Points O, P, R, S; and join EO, OF, FP, PG, GR, RH, HS, SE; then each of the Triangles EOF, FPG, GRH, HSE, is greater than one half of the Segment of the Circle EFGH, in which it is; and erect a Pyramid upon each of the Triangles EOF, FPG, GRH, HSE, having the ſame Altitude as the Cone: Then each of the Pyramids, thus erected, is greater than half its correſponding Segment of the Cone; wherefore, biſecting the remaining Circumferences, joining Right Lines, and erecting Pyramids upon each of the Triangles, having the ſame Vertex as the Cone; and doing this continually, we ſhall leave, at laſt, certain Segments of the Cone, that ſhall be leſs than the Exceſs by which the Cone EFGHN exceeds the Solid X. Let theſe be the Segments that ſtand on EO, OF, FP, PG, GR, RH, HS, SE; then the remaining Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, is greater than the Solid X: Alſo, let the Polygon ATBYCVDQ be deſcribed in the Circle ABCD, ſimilar and alike ſituate to the Polygon EOFPGRHS; upon which erect a Pyramid having the ſame Altitude as the Cone; and let LBT be one of the Triangles containing the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L; as likewiſe NFO one of the Triangles containing the Pyramid EOFPGRHS, and Vertex the Point N; and let KT, MO, be joined: Then, becauſe the Cone ABCDL is ſimilar to the Cone EFGHN, it ſhall be, as BD is to FH, ſo is the Axis KL to the Axis MN: But as BD is to FH,
* 15. 5. ſo is * BK to FM; conſequently, as BK is to FM, ſo is KL to MN; and (by Alternation) as BK is to KL, ſo is FM to MN. And ſince each is perpendicular, and the Sides about the equal Angles BKL, FMN are proportional; the Triangle BKL ſhall be
† 6. 6. † ſimilar to the Triangle FMN. Again, becauſe BK is to KT, as FM is to MO, the Sides are proportional about the equal Angles BKT, FMO, (for the Angle BKT is the ſame Part of the four Right Angles at the Centre K, as the Angle FMO is of the four Right Angles

Angles at the Centre M); therefore the Triangle BKT shall be * similar to the Triangle FMO. And * 6. 6.
because is has been proved, that BK is to KL, as FM is to MN; and BK is equal to KT; and FM to MO; it shall be, as TK is to KL, so is OM to MN; and the proportional Sides are about the equal Angles TKL, OMN; for they are Right Angles: Therefore the Triangle LKT shall be similar to the Triangle MNO. And since, by the Similarity of the Triangles BKL, FMN, it is, as LB is to BK, so is NF to FM; and, by the Similarity of the Triangles BKT, FMO, it is, as KB is to BT, so is MF to FO; it shall be (by Equality of Proportion), as LB is to BT, so is NF to FO. Again, since, by the Similarity of the Triangles LTK, NOM, it is, as LT is to TK, so is NO to OM; and, by the Similarity of the Triangles KBT, OMF, it is, as KT is to TB, so is MO to OF; it shall be (by Equality of Proportion), as LT is to TB, so is NO to OF. But it has been proved, that TB is to BL, as OF is to FN; wherefore, again (by Equality of Proportion), as TL is to LB, so is ON to NF; and therefore the Sides of the Triangles LTB, NOF, are proportional; and so the Triangles LTB, NOF, are equiangular and similar to each other; and, consequently, the Pyramid, whose Base is the Triangle BKT, and Vertex the Point L, is similar to the Pyramid whose Base is the Triangle FMO, and Vertex the Point N; for they are contained under similar Planes equal in Multitude: But similar Pyramids that have triangular Bases, are † to one another in the triplicate † 8 of this.
Proportion of their homologous Sides; therefore the Pyramid BKTL to the Pyramid FMON, has a triplicate Proportion of that which BK has to FM. In like manner, drawing Right Lines from the Points A, Q, D, V, C, Y, to K; as also others from the Points E, S, H, R, G, P, to M; and erecting Pyramids on the Triangles having the same Vertices as the Cones, we demonstrate, that every Pyramid of one Cone, to every one of the other Cone, has a triplicate Proportion of that which the Side BK has to the homologous Side MF, that is, which BD has to FH. But as one of the Antecedents is to one of the Consequents, so are ‡ all ‡ 12. 5.
the Antecedents to all the Consequents. Therefore, as

the Pyramid BKTL is to the Pyramid FMON, ſo is the whole Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the whole Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N. Wherefore the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, has a triplicate Proportion of that which BD hath to FH. But the Cone whoſe Baſe is the Circle ABCD, and Vertex the Point L, is ſuppoſed to have to the Solid X a triplicate Proportion of that which BD has to FH; therefore as the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, is to the Solid X, ſo is the Pyramid whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N. But the ſaid Cone is greater than the Pyramid that is in it, for it comprehends it; therefore the Solid X alſo is greater than the Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N; but it is alſo leſs, which is abſurd. Therefore the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, to ſome Solid leſs than the Cone, whoſe Baſe is the Circle EFGH, and Vertex the Point N, has not a triplicate Proportion of that which BD has to FG. In like manner we demonſtrate, that the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, has not a triplicate Proportion of that which FH has to BD. Laſtly, I ſay, the Cone ABCDL, to a Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH: For, if this be poſſible, let it be ſo to ſome Solid Z greater than the Cone EFGHN; then (by Inverſion) the Solid Z, to the Cone ABCDL, has a triplicate Proportion of that which FH has to BD. But ſince the Solid Z is greater than the Cone EFGHN, the Solid Z ſhall be to the Cone ABCDL, as the Cone EFGHN is to ſome Solid leſs than the Cone ABCDL; and therefore the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, hath a triplicate Proportion of that which EA has to BD, which has been proved to be impoſſible; therefore the Cone ABCDL, to ſome Solid greater than the

the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH. It has been also demonstrated, that the Cone ABCDL, to some Solid less than the Cone EFGHN, hath not a triplicate Proportion of that which BD has to FH; wherefore the Cone ABCDL, to the Cone EFGHN, has a triplicate Proportion of that which BD has to FH. But as Cone is to Cone, so is * Cylinder to Cylinder; for a Cylinder, having the same Base as a Cone, and the same Altitude, is † triple of the Cone; since it is demonstrated, that every Cone is one third Part of a Cylinder, having the same Base, and equal Altitude: Therefore, also, a Cylinder to a Cylinder has a triplicate Proportion of that which BD has to FH. Therefore, *similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Bases*; which was to be demonstrated.

* 15. 5.

† 10 of this.

PROPOSITION XIII.

THEOREM.

If a Cylinder be divided by a Plane parallel to the opposite Planes; then, as one Cylinder is to the other Cylinder, so is the Axis to the Axis.

LET the Cylinder AD be divided by the Plane GH, parallel to the opposite Planes AB, CD, and meeting the Axis EF in the Point K. I say, as the Cylinder BG is to the Cylinder GD, so is the Axis EK to the Axis KF.

For, let the Axis EF be both Ways produced to L and M; and put any Number of Lines EN, NL, &c. each equal to the Axis EK; and any Number of Lines FX, XM, &c. each equal to FK; and thro' the Points L, N, X, M, let Planes parallel to AB, CD, pass; and in those Planes from L, N, X, M, as Centres, describe the Circles OP, RS, TY, VQ, each equal to AB, CD; and conceive the Cylinders PR, RB, DT, TQ, to be compleated: Then, because the Axis LN, NE, EK, are equal to each other, the Cylinders PR, RB, BG, will be * to one another as their Bases; and therefore the Cylinders PR, RB, BG, are equal: And since the Axis LN, NE, EK, are equal to each other;

* 11 of this.

as

as alſo the Cylinders PR, RB, BG; and the Number of Lines LN, NE, EK, is equal to the Number of Lines PR, RB, BG; the Axis KL ſhall be the ſame Multiple of the Axis EK, as the Cylinder PG is of the Cylinder GB. For the ſame Reaſon, the Axis MK is the ſame Multiple of the Axis KF, as the Cylinder GQ is of the Cylinder GD. Now, if the Axis KL be equal to the Axis KM, the Cylinder PG ſhall be equal to the Cylinder GQ; if the Axis KL be greater than the Axis KM, the Cylinder PG ſhall be likewiſe greater than the Cylinder GQ; and, if leſs, leſs: Therefore, becauſe there are four Magnitudes, *viz.* the Axis EK, KF, and the Cylinders BG, GD; and there are taken their Equimultiples, namely, the Axis KL, and the Cylinder PG, the Equimultiples of the Axis EK, and the Cylinder BG; and the Axis KM, and the Cylinder GQ, the Equimultiples of the Axis KF, and the Cylinder GD: And it is demonſtrated, that if the Axis KL exceeds the Axis KM, the Cylinder PG will exceed the Cylinder GQ; and, if it be equal, equal; and, if leſs, leſs. Therefore, as the Axis EK is to the Axis KF, ſo † is the Cylinder BG to the Cylinder GD. Wherefore, *if a Cylinder be divided by a Plane parallel to the oppoſite Planes; then, as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis*; which was to be demonſtrated.

† *Def.* 5. 5.

PROPOSITION XIV.

THEOREM.

Cones and Cylinders, being upon equal Baſes, are to one another as their Altitudes.

LET the Cylinders EB, FD, ſtand upon equal Baſes AB, CD. I ſay, as the Cylinder EB is to the Cylinder FD, ſo is the Axis GH to the Axis KL.

For, produce the Axis KL to the Point N; and put LN equal to the Axis GH; and let a Cylinder CM be conceived about the Axis LN: Then, becauſe the Cylinders EB, CM, have the ſame Altitude, they are * to one another as their Baſes. But their Baſes are equal; therefore the Cylinders EB, CM, will be alſo equal. And becauſe the Cylinder

* 11 *of this*

FM

FM is cut by a Plane CD, parallel to the oppofite Planes, it fhall be as the Cylinder CM is to the Cylinder FD, fo is the Axis LN to the Axis KL. But the Cylinder CM is equal to the Cylinder EB; and the Axis LN to the Axis GH; therefore the Cylinder EB is to the Cylinder FD, as the Axis GH is to the Axis KL: And as the Cylinder EB is to the Cylinder FD, fo is ‡ the Cone ABG to the Cone CDK; for the Cylinders are * triple of the Cones. Therefore, as the Axis GH is to the Axis KL, fo is the Cone ABG to the Cone CDK; and fo the Cylinder EB to the Cylinder FD. Wherefore, *Cones and Cylinders, being upon equal Bafes, are to one another as their Altitudes*; which was to be demonftrated. ‡ 15. 5. * 10 *of this.*

PROPOSITION XV.

THEOREM.

The Bafes and Altitudes of equal Cones and Cylinders are reciprocally proportional; and Cones and Cylinders, whofe Bafes and Altitudes are reciprocally proportional, are equal to one another.

LET the Bafes of the equal Cones and Cylinders be the Circles ABCD, EFGH, and their Diameters AC, EG; and Axis KL, MN; which are alfo the Altitudes of the Cones and Cylinders: And let the Cylinders AX, EO, be compleated. I fay, the Bafes and Altitudes of the Cylinders AX, EO, are reciprocally proportional; that is, the Bafe ABCD is to the Bafe EFGH, as the Altitude MN is to the Altitude KL.

For, the Altitude KL is either equal to the Altitude MN, or not equal. Firft, let it be equal; and the Cylinder AX is equal to the Cylinder EO. But Cylinders and Cones, that have the fame Altitude, are * to one another as their Bafes; therefore the Bafe ABCD is equal to the Bafe EFGH: And confequently, as the Bafe ABCD is to the Bafe EFGH, fo is the Altitude MN to the Altitude KL. But if the Altitude KL be not equal to the Altitude MN, let MN be the greater; and take PM, equal to LK, from MN; * 11 *of this.*

MN; and let the Cylinder EO be cut thro' P by the Plane TYS, parallel to the oppofite Planes of the Circles EFGH, RO; and conceive ES to be a Cylinder, whofe Bafe is the Circle EFGH, and Altitude PM: Then, becaufe the Cylinder AX is equal to the Cylinder EO, and ES is fome other Cylinder; the Cylinder AX to the Cylinder ES, fhall be as the Cylinder EO is to the Cylinder ES. But as the Cylinder

* 11 of this. AX is to the Cylinder ES, fo is * the Bafe ABCD to the Bafe EFGH; for the Cylinders AX, ES, have the fame Altitude: And as the Cylinder EO is to the Cy-

† 13 of this. linder ES, fo is † the Altitude MN to the Altitude MP; for the Cylinder EO is cut by the Plane TYS, parallel to the oppofite Planes. Therefore as the Bafe ABCD is to the Bafe EFGH, fo is the Altitude MN to the Altitude MP. But the Altitude MP is equal to the Altitude KL; wherefore, as the Bafe ABCD is to the Bafe EFGH, fo is the Altitude MN to the Altitude KL; and therefore, *the Bafes and Altitudes of the equal Cylinders* AX, EO, *are reciprocally proportional.*

And if the Bafes and Altitudes of the Cylinders AX, EO, are reciprocally proportional; that is, if the Bafe ABCD be to the Bafe EFGH, as the Altitude MN is to the Altitude KL; I fay, the Cylinder AX is equal to the Cylinder EO. For, the fame Conftruction remaining, becaufe the Bafe ABCD is to the Bafe EFGH, as the Altitude MN is to the Altitude KL; and the Altitude KL is equal to the Altitude MP; it fhall be, as the Bafe ABCD is to the Bafe EFGH, fo is the Altitude MN to the Altitude MP. But as the Bafe ABCD is to the Bafe EFGH, fo is the Cylinder AX to the Cylinder ES, for they have the fame Altitude; alfo, as the

‡ 13 of this. Altitude MN is to the Altitude MP, fo is ‡ the Cylinder EO to the Cylinder ES. Therefore, as the Cylinder AX is to the Cylinder ES, fo is the Cylinder EO to the Cylinder ES: Wherefore, *the Cylinder* AX *is equal to the Cylinder* EO; which was to be demonftrated.

In like manner we prove this in Cones.

PROPOSITION XVI.

PROBLEM.

Two Circles being about the ſame Centre, to inſcribe in the greater a Polygon of equal Sides, even in Number, that ſhall not touch the leſſer Circle.

LET ABCD, EFGH, be two given Circles about the Centre K; it is required to inſcribe a Polygon of equal Sides, even in Number, in the Circle ABCD, not touching the leſſer Circle EFGH.

Draw the Right Line BD thro' the Centre K, as alſo AG, from the Point G, at Right Angles to BD, which produce to C; this Line will * touch the Circle EFGH: Then, biſecting the Circumference BAD, and again biſecting the Half thereof, and doing this continually, we ſhall have a Circumference left, at laſt, leſs than AD ‡. Let this Circumference be LD, and draw LM, from the Point L, perpendicular to BD, which produce to N; and join LD, DN: And then LD is † equal to DN. And ſince LN is parallel to AC, and AC touches the Circle EFGH, LN will not touch the Circle EFGH; and much leſs do the Right Lines LD, DN, touch, the Circle. And if Right Lines, each equal to LD, be applied round the Circle ABCD, *we ſhall have a Polygon inſcribed therein of equal Sides, even in Number, that does not touch the leſſer Circle* EFGH; which was to be done.

* 16. 3.

‡ Lemma.

† 29. 3.

PROPOSITION XVII.

PROBLEM.

To deſcribe a ſolid Polyhedron, in the greater of two Spheres, having the ſame Centre, which ſhall not touch the Superficies of the leſſer Sphere.

LET two Spheres be ſuppoſed about the ſame Centre A; it is required to deſcribe a ſolid Polyhedron in the greater Sphere, not touching the Superficies of the leſſer Sphere.

Let

Let the Spheres be cut by some Plane passing thro' the Centre; then the Sections will be Circles: For, because a Sphere is * made by the turning of a Semicircle about the Diameter, which is at Rest; in whatsoever Position the Semicircle is conceived to be, the Plane in which it is shall make a Circle in the Superficies of the Sphere. It is also manifest, that this Circle is a greater Circle, since the Diameter of the Sphere, which is likewise the Diameter of the Semicircle, is † greater than all Right Lines that are drawn in the Circle or Sphere. Now, let BCDE be that Circle of the greater Sphere, and FGH of the lesser Sphere; and let BD, CE, be two of their Diameters drawn at Right Angles to one another; let BD meet the lesser Circle in the Point G, and let GL be drawn at Right Angles to AG, and AL be joined: Then, bisecting the Circumference EB, as also the Half thereof, and doing thus continually, we shall have left, at last, a certain Circumference less than that Part of the Circumference of the Circle BL, which is subtended by a Right Line equal to GL. Let this be the Circumference BK; then the Right Line BK is less than GL; and BK shall be the * Side of a Polygon of equal Sides, even in Number, not touching the lesser Circle: Now, let the Sides of the Polygon, in the Quadrant of the Circle BE, be the Right Lines BK, KL, DM, ME; and produce the Line joining the Points K, A, to N; and raise ‡ AX from the Point A, perpendicular to the Plane of the Circle BCDE, meeting the Superficies of the Sphere in the Point X; and let Planes be drawn thro' AX and BD, and thro' AX and KN; which, from what has been said, will make great Circles in the Superficies of the Sphere; and let BXD, KXN, be Semicircles on the Diameters BD, KN: Then, because XA is perpendicular to the Plane of the Circle BCDE, all Planes that pass thro' XA shall also * be perpendicular to that same Plane. Therefore the Semicircles BXD, KXN, are perpendicular to that same Plane. And because the Semicircles BED, BXD, KXN, are equal; for they stand upon equal Diameters BD, KN; their Quadrants BE, BX, KX, shall be also equal. And therefore, as many Sides as the Polygon in the Quadrant BE has, so many Sides may there be

* Def. 14. 11.

† 15. 3.

* 16 of this.

‡ 12. 11.

* 18. 11.

be in the Quadrants BX, KX, equal to the Sides BK, KL, LM, ME. Let thoſe Sides be BO, OP, PR, RX, KS, ST, TY, YX; and join SO, TP, YR; and let Perpendiculars be drawn from O, S, to the Plane of the Circle BCDE: Theſe will † fall on BD, KN, the common Sections of the Planes; becauſe the Planes of the Semicircles BXD, KXN, are perpendicular to the Plane of the Circle BCDE: Let the ſaid Perpendicular be OV, SQ, and join VQ; then, ſince the equal Circumferences BO, SK, are taken in the equal Semicircles BXD, KXN, and OV, SQ, are Perpendiculars; OV ſhall be equal to SQ, and BV to KQ. But the Whole BA is equal to the Whole KA; therefore the Part remaining VA is equal to the Part remaining QA: Therefore, as BV is to VA, ſo is KQ to QA; and ſo VQ is ‡ parallel to BK. And ſince OV and SQ are both perpendicular to the Plane of the Circle BCDE, OV ſhall be * parallel to SQ. But it has alſo been proved equal to it; wherefore QV, SO, are † equal and parallel. And becauſe QV is parallel to SO, and alſo parallel to KB; SO ſhall be alſo ‡ parallel to KB: Join BO, KS, and then, KBOS is * a quadrilateral Figure in one Plane: For if two Right Lines be parallel, and Points be taken in both of them, a Right Line joining the ſaid Points is in the ſame Plane as the Parallels are. And for the ſame Reaſon, each of the quadrilateral Figures SOPT, TPRY, are in one Plane. And the Triangle YRX is † in one Plane; therefore, if Right Lines be ſuppoſed to be drawn from the Points O, S, P, T, R, Y, to the Point A, there will be conſtituted a certain ſolid polyhedrous Figure within the Circumferences BX, KX, compoſed of Pyramids, whoſe Baſes are the quadrilateral Figures KBOS, SOPT, TPRY, and the Triangle YRX; and Vertices the Point A. And if there be made the ſame Conſtruction on each of the Sides KL, LM, ME, like as we have done on the Side KB; and alſo in the other three Quadrants, and the other Hemiſphere, there will be conſtituted a polyhedrous Figure deſcribed in the Sphere, compoſed of Pyramids whoſe Baſes will be equal and ſimilar to the aforeſaid quadrilateral Figures, and Triangle YRX, and Vertices

† 38. 11.
‡ 2. 6.
* 6. 11.
† 33. 1.
‡ 9. 11.
* 7. 11.
† 2. 11.

tices

tices the Point A. Now, I say, the said Polyhedron does not touch the Superficies of the Sphere, wherein the Circle FGH is. Let AZ be drawn ‡ from the Point A, perpendicular to the Plane of the quadrilateral Figure KBSO, meeting it in the Point Z; and join BZ, ZK: Then, since AZ is perpendicular to the Plane of the quadrilateral Figure KBSO, it shall also be * perpendicular to all Right Lines that touch it, and are in the same Plane: Wherefore AZ is perpendicular to BZ and ZK. And because AB is equal to AK, the Square of AB shall be also equal to the Square of AK; and the Squares of AZ, ZB, are † equal to the Square of AB; for the Angle at Z is a Right Angle; and the Squares of AZ, ZK, are equal to the Square of AK: Therefore the Square of AZ, ZB, are equal to the Squares of AZ, ZK. Let the common Square of AZ be taken away, and then the Square of BZ, remaining, is equal to the Square of ZK, remaining; and so the Right Line BZ is equal to the Right Line ZK. After the same manner we demonstrate, that Right Lines drawn from the Point Z to the Points O, S, are each equal to BZ, ZK. Therefore a Circle described about the Centre Z, with either of the Distances, ZB, ZK, will also pass thro' the Points O, S. And, because BKSO is a quadrilateral Figure in a Circle, and OB, BK, KS, are equal; and OS is less than BK; the Angles BZK shall be obtuse; and so BK greater than BZ. But GL also is greater than BK; therefore GL is much greater than BZ; and the Square of GL is greater than the Square of BZ. And since AL is equal to AB, the Square of AL shall be equal to the Square of AB. But the Squares of AG, GL, together, are equal to the Square of AL; and the Squares of BZ, ZA, together, equal to the Square of AB: Therefore the Squares of AG, GL, together, are equal to the Squares of BZ, ZA, together. But the Square of BZ is less than the Square of GL; therefore the Square of ZA is greater than the Square of AG; and so the Right Line ZA will be greater than the Right Line AG. But AZ is perpendicular to one Base of the Polyhedron, and AG reaches to the Superficies of the lesser Sphere: Wherefore the Polyhedron does not touch the Superficies of the lesser Sphere. Therefore, *there is described a solid Polyhedron*

‡ 11. 11.

* Def. 3. 11.

† 4. 17.

in the greater of two Spheres, having the same Centre, which doth not touch the Superficies of the lesser Sphere; which was to be done.

Coroll. Also, if a solid Polyhedron be described in some other Sphere, similar to that which is described in the Sphere BCDE; the solid Polyhedron described in the Sphere BCDE, to the solid Polyhedron described in that other Sphere, shall have a triplicate Proportion of that which the Diameter of the Sphere BCDE hath to the Diameter of that other Sphere. For, the Solids being divided into Pyramids equal in Number, and of the same Order, the same Pyramids shall be similar. But similar Pyramids are to each other in a triplicate Proportion of their homologous Sides; therefore the Pyramid, whose Base is the quadrilateral Figure KBOS, and Vertex the Point A, to the Pyramid of the same Order in the other Sphere, has a triplicate Proportion of that which the homologous Side of the one has to the homologous Side of the other; that is, which AB, drawn from the Centre A of the Sphere, to that Line which is drawn from the Centre of the other Sphere. In like manner, every one of the Pyramids, that are in the Sphere whose Centre is A, to every one of the Pyramids of the same Order in the other Sphere, hath a triplicate Proportion of that which AB has to that Line drawn from the Centre of the other Sphere: And as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents. Wherefore the whole solid Polyhedron, which is in the Sphere described about the Centre A, to the whole solid Polyhedron that is in the other Sphere, hath a triplicate Proportion of that which AB hath to the Line drawn from the Centre of the other Sphere; that is, which the Diameter BD has to the Diameter of the other Sphere.

PRO-

PROPOSITION XVIII.

Spheres are to one another in a triplicate Proportion of their Diameters.

SUppose ABC, DEF, are two Spheres, whose Diameters are BC, EF. I say, the Sphere ABC to the Sphere DEF, has a triplicate Proportion of that which BC has to EF.

For, if it be not so, the Sphere ABC to a Sphere either lesser or greater than DEF, will have a triplicate Proportion of that which BC has to EF. First, let it be to a lesser, as GHK; and suppose the Sphere DEF to be described about the Sphere GHK; and let there be described * a solid Polyhedron in the greater Sphere DEF, not touching the Superficies of the lesser Sphere GHK; also, let a solid Polyhedron be described in the Sphere ABC, similar to that which is described in the Sphere DEF; then the solid Polyhedron in the Sphere ABC, to the solid Polyhedron in the Sphere DEF, will have † a triplicate Proportion of that which BC has to EF: But the Sphere ABC to the Sphere GHK, hath a triplicate Proportion of that which BC hath to EF; therefore, as the Sphere ABC is to the Sphere GHK, so is the solid Polyhedron in the Sphere ABC, to the solid Polyhedron in the Sphere DEF; and (by Inversion) as the Sphere ABC is to the solid Polyhedron that is in it, so is the Sphere GHK to the solid Polyhedron that is in the Sphere DEF. But the Sphere ABC is greater than the solid Polyhedron that is in it; therefore the Sphere GHK is also greater than the solid Polyhedron that is in the Sphere DEF, and also less than it, as being comprehended thereby, which is absurd; therefore the Sphere ABC to a Sphere less than the Sphere DEF, hath not a triplicate Proportion of that which BC has to EF. After the same manner it is demonstrated, that the Sphere DEF to a Sphere less than ABC, has not a triplicate Proportion of that which EF has to BC: I say, moreover, that the Sphere ABC to a Sphere greater than DEF, hath

* 17 *of this.*

† *Cor. to the last Prop.*

not

not a triplicate Proportion of that which BC has to EF: For, if it be poſſible, let it have to the Sphere LMN greater than DEF; then (by Inverſion) the Sphere LMN to the Sphere ABC, ſhall have a triplicate Proportion of that which the Diameter EF has to the Diameter BC. But as the Sphere LMN is to the Sphere ABC, ſo is the Sphere DEF to ſome Sphere leſs than ABC, becauſe the Sphere LMN is greater than DEF. Therefore the Sphere DEF to a Sphere leſs than ABC, hath a triplicate Proportion of that which EF has to BC, which is abſurd, as has been before proved. Therefore the Sphere ABC to a Sphere greater than DEF, has not a triplicate Proportion of that which BC has to EF. But it has alſo been demonſtrated, that the Sphere ABC to a Sphere leſs than DEF, has not a triplicate Proportion of that which BC has to EF: Therefore, *the Sphere* ABC *to the Sphere* DEF, *has a triplicate Proportion of that which* BC *has to* EF; which was to be demonſtrated.

FINIS.

THE

ELEMENTS

Of PLANE and SPHERICAL

TRIGONOMETRY.

DEFINITIONS.

THE Business of Trigonometry is, to find the Angles when the Sides are given, and the Sides, or the Ratios of the Sides, when the Angles are given; and to find Sides and Angles, when Sides and Angles are given: In order to which, it is necessary, that not only the Peripheries of Circles, but also certain Right Lines in and about Circles, be supposed divided into some determined Number of Parts.

And so the antient Mathematicians thought fit to divide the Periphery of a Circle into 360 *Parts, which they call Degrees; and every Degree into* 60 *Minutes; and every Minute into* 60 *Seconds; and, again, every Second into* 60 *Thirds; and so on. And every Angle is said to be of such a Number of Degrees and Minutes, as there are in the Arc measuring that Angle.*

There are some that would have a Degree divided into centesimal Parts, rather than sexagesimal ones; and perhaps it would be more useful to divide, not only a Degree, but even the whole Circle, into a d[illegible] Ratio; which Division may some Time or other gain Place. Now, if a Circle contains 360 *Degrees, a Quadrant thereof, which is the Measure of a Right Angle, will*

 be

be 90 of those Parts: And if it contains 100 Parts, a Quadrant will be 25 of these Parts.

The Complement of an Arc is the Difference thereof from a Quadrant.

A Chord or Subtense, is a Right Line drawn from one End of the Arc to the other.

The Right Sine of any Arc, which is also commonly called only a Sine, is a Right Line drawn, from one End of an Arc, perpendicular to the Radius drawn thro' the other End of the said Arc; and is therefore the Semisubtense of double the Arc; viz. DE $=\frac{1}{2}$ DO, and the Arc DO is double of the Arc DB. Hence, the Sine of an Arc of 30 Degrees is equal to one half of the Radius. For (by Corol. 15. El. 4.) the Side of an Hexagon inscribed in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius. A Sine divides the Radius into two Segments CE, EB; one of which, CE, which is intercepted between the Centre and the Right Sine, is the Sine of the Complement of the Arc DB to a Quadrant (for CE $=$ FD, which is the Sine of the Arc DH), and is called the Cosine: The other Segment BE, which is intercepted between the Right Sine and the Periphery, is called a Versed Sine, and sometimes a Sagitta.

And if the Right Line CG be produced from the Centre C, thro' one End D of the Arc, until it meets the Right Line BG, which is perpendicular to the Diameter thro' the other End B of the Arc; then CG is called the Secant, and BG the Tangent, of the Arc DB.

The Cosecant and Cotangent of an Arc are the Secant and Tangent of that Arc which is the Complement of the former Arc to a Quadrant. Note, As the Chord of an Arc, and of its Complement to a Circle, is the same; so, likewise, are the Sine, Tangent, and Secant, of an Arc, the same as the Sine, Tangent, and Secant, of its Complement to a Semicircle.

The Sinus Totus is the greatest Sine, or the Sine of 90 Degrees, which is equal to the Radius of the Circle.

A Trigonometrical Canon is a Table, which, beginning from one Minute, orderly expresses the Lengths that every Sine, Tangent, and Secant have, in respect of the Radius, which is supposed Unity; and is conceived to be divided into 10,000,000 or more decimal Parts. And so the Sine, Tangent, or Secant, of an Arc, may be had

by

by Help of this Table; and, contrariwise, a Sine, Tangent, or Secant, being given, we may find the Arc it expresses. Take notice, That in the following Tract, R. signifies the Radius, S. a Sine, Cos. a Cosine, T. a Tangent, and Cot. a Cotangent; also ACq signifies the Square of the Right Line AC; and the Marks or Characters +, —, =, : : :, and √, are, severally, used to signify Addition, Subtraction, Equality, Proportionality, and the Extraction of the Square Root: Again, when a Line is drawn over the Sum or Difference of two Quantities, then that Sum or Difference is to be considered as one Quantity.

The CONSTRUCTIONS of the *Trigonometrical Canon*.

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is also given.

FOR (by 47 of the first *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$, and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the Square Root, there is given $AC = \sqrt{\overline{ABq + BCq}}$; and $AB = \sqrt{\overline{ACq - BCq}}$; and $BC = \sqrt{\overline{ACq - ABq}}$.

PROPOSITION II.

PROBLEM.

The Sine DE *of the Arc* BD, *and the Radius* CD, *being given, to find the Cosine* DF.

THE Radius CD, and the Sine DE, being given in the Right-angled Triangle CDE, there will be given (by the last *Prop.*) $\sqrt{\overline{CDq - DEq}} = (CE =)$ DF.

PROPOSITION III.

PROBLEM.

The Sine DE *of any Arc* DB *being given, to find* DM *or* BM, *the Sine of Half the Arc.*

DE being given, CE (by the last *Prop.*) will be given, and accordingly EB, which is the Difference between the Cosine and Radius. Therefore DE, EB, being given, in the Right-angled Triangle DBE, there will be given DB, whose Half DM is the Sine of the Arc DL $= \frac{1}{2}$ the Arc BD.

PROPOSITION IV.

PROBLEM.

The Sine BM *of the Arc* BL *being given, to find the Sine of double that Arc.*

THE Sine BM being given, there will be given (by *Prop.* 2.) the Cosine CM. But the Triangles CBM, DBE, are equiangular, because the Angles at E and M are Right Angles, and the Angle at B common: Wherefore (by 4. 6.) we have CB : CM : : (BD, or) 2 BM : DE. Whence, since the three first Terms of this Analogy are given, the fourth also, which is the Sine of the Arc DB, will be known.

Coroll. Hence CB : 2 CM : : BD : 2 DE; that is, the Radius is to double the Cosine of one Half of the Arc DB, as the Subtense of the Arc DB is to the Subtense of double that Arc. Also, CB : 2 CM : : (2 BM : 2 DE : :) BM : DE : : $\frac{1}{2}$ CB : CM. Wherefore the Sine of an Arc, and the Sine of its Double, being given, the Cosine of the Arc itself is given.

PRO-

ROPOSITION V.

ROBLEM.

The Sines of two Arcs, BD, FD, being given, to find FI the Sine of the Sum, as likewise EL, the Sine of their Difference.

LET the Radius CD be drawn, and then CO is the Cosine of the Arc FD, which accordingly is given, and draw OP thro' O parallel to DK; also let OM, GE, be drawn parallel to CB: Then, because the Triangles CDK, COP, CHI, FOH, FOM, are equiangular; in the first Place CD : DK : : CO : OP, which, consequently, is known. Also, we have CD : CK : : FO : FM; and so, likewise, this shall be known. But because FO=EO, then will FM=MG=ON; and so OP + FM = FI = Sine of the Sum of the Arcs: And OP—FM; that is, OP—ON=EL= Sine of the Difference of the Arcs; which were to be found.

Coroll. Because the Differences of the Arcs BE, BD, BF, are equal, the Arc BD shall be an Arithmetical Mean between the Arcs BE, BF.

PROPOSITION VI.

THEOREM.

The same Things being supposed, the Radius is to double the Cosine of the mean Arc, as the Sine of the Differences is to the Difference of the Sines of the Extremes.

FOR we have CD : CK : : FO : FM; whence, by doubling the Consequents, CD : 2 CK : : FO : (2 FM, or) to FG, which is the Difference of the Sines EL, FI. W. W. D.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI, EL, shall be equal to the Sine FO, of the Difference. For, in this Case, CK is the Sine of 30 Degrees; the double whereof is equal to the Radius; and so, since CD=2 CK, we shall have FO

FO=FG. And, consequently, if the two Arcs BE, BF, are equidistant from the Arc of 60 Degrees, the Difference of the Sines shall be equal to the Sine of the Difference FD.

Coroll. 2. Hence, if the Sines of all Arcs distant from one another by a given Interval, be given, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees=the Sine of 59 Degrees+the Sine of 1 Degree; and the Sine of 62 Degrees=the Sine of 58 Degrees+the Sine of 2 Degrees: Also, the Sine of 63 Degrees=the Sine of 57 Degrees+the Sine of 3 Degrees, and so on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, distant from each other by a given Interval, be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the precedent Analogy, all the Sines to 30 Degrees may be found: For the Radius is to double the Cosine of 15 Degrees, as the Sine of 1 Degree is to the Difference of the Sines of 14 Degrees, and 16 Degrees: So, also, is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and so on continually, until you come to the Sine of 30 Degrees.

After the same manner, as the Radius is to double the Cosine of 30 Degrees, or to double the Sine of 60 Degrees, so is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees : : Sine 2 Degrees to the Difference of the Sines of 28 and 32 Degrees : : Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But, in this Case, the Radius is to double the Cosine of 30 Degrees, as to $\surd$ 3*.

See FIG. *of the* DEFINITIONS.

* Let BD be an Arc of 30 Degrees:

Rad. Tan. Cosine Sine

Then, as CB : BG : : FD : DE. DO=CB; *ergo* DE$=\frac{1}{2}$; $\sqrt{CDq - DEq} = CE = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}}$; CD : CE : : 1 $\sqrt{\frac{3}{4}}$; CD : 2CE : : 1 : $2\sqrt{\frac{3}{4}} = \sqrt{\frac{12}{4}}\sqrt{3}$. *Q. E. D.*

And,

And, accordingly, if the Sines of the Distances from the Arc of 30 Degrees, be multiplied by $\sqrt{3}$, the Difference of the Sines will be had.

So, likewise, may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Cosines of one and two Minutes given. For, as the Radius is to double the Cosine of 2' : : Sine 1' : Difference of the Sines of 1' and 3' : : Sine 2' : Difference of the Sines of 0' and 4'; that is, to the Sine of 4'. And so, the Sines of the four first Minutes being given, we may thereby find the Sines of the others to 8', and from thence to 16', and so on.

PROPOSITION VII.

THEOREM.

In small Arcs, the Sines and Tangents of the same Arcs are nearly to one another, in a Ratio of Equality.

FOR, because the Triangles CED, CBG, are equiangular, CE : CB : : ED : BG. But as the Point E approaches B, EB will vanish in respect of the Arc BD; whence CE will become nearly equal to CB, and so ED will be also nearly equal to BG. If EB be less than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be also less than the $\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is less than the Tangent, and greater than its Sine, and the Sine and Tangent of a very small Arc are nearly equal; it follows, that the Arc shall be nearly equal to its Sine: And so, in very small Arcs, it shall be, as Arc is to Arc, so is Sine to Sine.

PRO-

PROPOSITION VIII.

Proble

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inscribed in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius (*by Coroll.* 15*th of the* 4*th*); and so the Half of the Radius shall be the Sine of the Arc of 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found (*by Prop.* 3.) Also the Sine of the Arc of 15 Degrees being given (*by the same Prop.*), we may have the Sine of 7 Degrees 30 Minutes: So, likewise, can we find the Sine of the Half of this, *viz.* 3 Degrees 45′; and so on, until twelve Bisections being made, we come to an Arc of $52^{2}, 44^{3}, 03^{4}, 45^{5}$, whose Cosine is nearly equal to the Radius; in which Case (*as is manifest from Prop.* 7.) Arcs are proportional to their Sines: And so, as the Arc of $52^{2}, 44^{3}, 03^{4}, 45^{5}$, is to an Arc of one Minute, so shall the Sine before found be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop.* 2. *and* 4.) the Sine and Cosine of two Minutes will be had.

PROPOSITION IX.

Theorem.

If the Angle BAC, *being in the Periphery of a Circle, be bisected by the Right Line* AD, *and if* AC *be produced until* DE=AD *meets it in* E; *then shall* CE=AB.

IN the quadrilateral Figure ABDC (*by* 22. 3.) the Angles B and DCA are equal to two Right Angles =DCE+DCA (*by* 13. 1.): Whence the Angle B =DCE. But, likewise, the Angle E=DAC (*by* 5. 1.)=DAB, and DC=DB: Wherefore the Triangles BAD and CED are congruous, and so CE is equal to AB. W. W. D.

PRO-

PROPOSITION X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, *&c. be equal; and let the Subtenses of the Arcs* AB, AC, AD, AE, *&c. be drawn; then will* AB : AC : : AC : AB+AD : : AD : AC+AE : : AE : AD+AF : : AF : AE+AG.

LET AD be produced to H, AE to I, AF to K, and AG to L, so that the Triangles ACH, ADI, AEK, AFL, be Isosceles ones: Then, because the Angle BAD is bisected, we shall have DH=AB (*by the last Prop.*); so likewise shall EI=AC, FK=AD, also GL=AE.

But the Isosceles Triangles ABC, ACH, ADI, AEK, AFL, because of the equal Angles at the Bases are equiangular: Wherefore it shall be, as AB : AC : : AC : (AH=) AB+AD : : AD : (AI=) AC+AE : : AE : (AK=) AD+AF : : AF : (AL=) AE+AG. W. W. D.

Coroll. 1. Because AB is to AC, as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB, it shall also be (*by Coroll. Prop.* 4.) as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB, so is $\frac{1}{2}$ AB : $\frac{1}{2}$ AC : : $\frac{1}{2}$ AC : $\frac{1}{2}$ AB + $\frac{1}{2}$ AD : : $\frac{1}{2}$ AD : $\frac{1}{2}$ AC+$\frac{1}{2}$ AE : : $\frac{1}{2}$ AE : $\frac{1}{2}$ AD+$\frac{1}{2}$ AF, *&c.* Now let each of the Arcs AB, BC, CD, *&c.* be 2′; then will $\frac{1}{2}$ AB be the Sine of one Minute, $\frac{1}{2}$ AC the Sine of 2 Minutes, $\frac{1}{2}$ AD the Sine of 3 Minutes, $\frac{1}{2}$ AE the Sine of 4 Minutes, *&c.* Whence, if the Sines of one and two Minutes be given, we may easily find all the other Sines, in the following manner.

Let the Cosine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. 59′, be called Q; and make the following Analogies; R. : 2 Q : : Sin. 2′ : S. 1′+S. 3′. + Wherefore the Sine of 3 Minutes will be given. Also, R. : 2 Q : : S. 3′ : S. 2′ S. 4′. Wherefore the S. 4′ is given. And R. : 2 Q : : S. 4′: S. 3′+5′; and so the Sine of 5′ will be had.

Like-

Likewise, R. : 2 Q : : S. 5′ : S. 4′+S. 6′; and so we shall have the Sine of 6′. And, in like manner, the Sines of every Minute of the Quadrant will be given. And because the Radius, or the first Term of the Analogy, is Unity, the Operations will be with great Ease and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only (*by Cor.* 1. *Prop.* 6.)

The Sines being given, the Tangents and Secants may be found from the following Analogies (in the Figure for the Definitions); because the Triangles CED, CBG, CHI, are equiangular, we have

CE : ED : : CB : BG; that is, Cos. : S. : : R. : T.

GB : BC : : CH : HI; that is, T. : R : : R. : Cot.

CE . CD : : CB : CG; that is, Cos. R. : : R. : Secant,

DE : CD : : CH : CI; that is, S. : R. : : R. : Cosec.

SCHOLIUM.

That great Geometrician, and incomparable Philosopher, Sir Isaac Newton, *was the first that laid down a Series converging* in infinitum; *from which, having the Arcs given, their Sines may be found. Thus, if an Arc be called* A, *and the Radius be an Unit, the Sine thereof will be found to be*

$$A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9} - \&c.$$

And the Cosine,

$$1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8} - \&c.$$

These Series in the Beginning of the Quadrant, when the Arc A *is but small, soon converge. For in the Series for the Sine, if* A *does not exceed* 10 *Minutes, the two first Terms thereof,* viz. $A - \frac{1}{6}A^3$, *give the Sine to* 15 *Places of Figures. If the Arc* A *be not greater than one Degree, the three first Terms will exhibit the Sine to* 15 *Places of Figures; and so the said Series are very useful for finding the first and last Sines of the Quadrant. But the greater the Arc* A *is, the more are the Terms of the Series required to have the Sine, in*

Num-

Numbers, true to a given Place of Figures. And then, when the Arc is nearly equal to the Radius, the Series converges very slow, and therefore, to remedy this, I have devised other Series, similar to the Newtonian *ones, wherein I suppose, the Arc, whose Sine is sought, is the Sum or Difference of two Arcs,* viz. $A+z$, *or* $A-z$: *And let the Sine of the Arc* A *be called* a, *and the Cosine* b. *Then the Sine of the Arc* $A+z$ *will be expressed thus:*

$$1.\quad a+\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\text{\&c.}$$

And the Cosine is

$$2.\quad b-\frac{az}{1}-\frac{bz^2}{1.2}+\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}-\frac{az^5}{1.2.3.4.5}-\frac{bz^6}{1.2.3.4.5.6}$$

In like manner the Sine of the Arc $A-z$ *is*

$$3.\quad a-\frac{bz}{1}-\frac{az^2}{1.2}+\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}-\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}$$

And the Cosine is

$$4.\quad b+\frac{az}{1}-\frac{bz^2}{1.2}-\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}+\frac{az^5}{1.2.3.4.5}\text{\&c.}$$

The Arc A, *is an arithmetical Mean between the Arc* $A-z$, *and the Arc* $A+z$. *And the Differences of the Sines are*

$$5.\quad \frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\text{\&c.}$$

$$6.\quad \frac{bz}{1}+\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}-\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}+\frac{az^6}{1.2.3.4.5.6}\text{\&c.}$$

Whence the Difference of the Differences, or second Difference, will be

$$7.\quad -\frac{2az^2}{1.2}-\frac{2az^4}{1.2.3.4}+\frac{2az^6}{1.2.3.4.5.6}\text{\&c.}$$

$$\text{Or } 2a\times\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}\text{\&c.}$$

Which

Which Series is equal to double the Sine of the mean Arc, drawn into the Versed Sine of the Arc z, *and converges very soon. So that if* z *be the first Minute of the Quadrant, the first Term of the Series gives the second Difference to 15 Places of Figures, and the second Term to 25 Places.*

From hence, if the Sines of the Arcs, distant one Minute from each other, be given; the Sines of all the Arcs, that are in the same Progression, may be found by an exceeding easy Operation.

In the first and second Series, if A=0; *then shall* a=0, *and* b *its Cosine will become Radius, or* 1. *And hence if the Terms wherein* a *is, are taken away, and* 1 *be put instead of* b, *the Series will become the* Newtonian. *In the third and fourth Series, if* A *be* 90 *Degrees, we shall have* b=0, *and* a=1. *Whence, again, taking away all the Terms wherein* b *is, and putting* 1 *instead of* a, *we shall have the* Newtonian *Series arise.*

Note, *All the said Series easily flow from the* Newtonian *ones. By the fifth* Proposition.

PROPOSITION XI.

THEOREM.

In a Right-angled Triangle, if the Hypothenuse be made the Radius, then are the Sides the Sines of their opposite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of its opposite Angle, and the Hypothenuse is the Secant of that Angle.

IT is manifest, that CB is the Sine of the Arc D, and AB the Cosine thereof; but the Arc CD is the Measure of the Angle A, and the Complement of the Measure of the Angle C: Moreover, if AB in the second Figure to this Proposition, be supposed Radius, then BC is the Tangent, and AC the Secant of the Arc BD, which is the Measure of the Angle A. So, also, if BC be made the Radius, then is BA the Tangent, and AC the Secant, of the Arc BE, or Angle C. W. W. D. Therefore, as AC, being taken as some given Measure, is to BC taken in the same Measure; so shall the Number 10,000,000 Parts, into which the Radius is supposed to be divided, be to a Number expressing, in the same

ſame Parts, the Length of the Sine of the Angle A; that is,

it will be, as AC : BC   R : S, A.
by the ſame Reaſon, as AC : BA   R : S, C.
alſo, as AB : BC   R : T, A.
and, as BC : BA   R : T, C.

And ſo, if any three of theſe Proportionals be given, the fourth may be found by the *Rule of Three.*

PROPOSITION XII.

THEOREM.

Sides of plane Triangles are as the Sines of their oppoſite Angles.

IF the Sides of a Triangle, inſcribed in a Circle, be biſected by perpendicular Radii; then ſhall the half Sides be the Sines of the Angles at the Periphery; for the Angle BDC, at the Centre, is double of the Angle BAC at the Periphery (*by* 20 *El. lib.* 3.); and ſo (the Half of BDC, *viz.*) BDE=BAC, and BE is the Sine of (BDE, or) BAC. For the ſame Reaſon, BF ſhall be the Sine of the Angle BCA, and AG the Sine of the Angle ABC.

In a Right-angled Triangle we have BD=½ BC =Radius (*by* 31 *Eucl.* 3.); but Radius is the Sine of a Right Angle: Whence half BC is the Sine of the Angle A.

In an obtuſe-angled Triangle, let BI, CI, be drawn; and then the Angle I ſhall be the Complement of the Angle A to two Right Angles (*by* 22 *El.* 3.); and ſo they ſhall both have the ſame Sine. But the Angle BDE (whoſe Sine is BE)=Angle I; therefore BE ſhall be the Sine of the Angle BAC. And ſo in every Triangle, the Halves of the Sides are the Sines of the oppoſite Angles: But it is manifeſt, that the Sides are to one another as their Halves. And therefore, *the Sides of plane Triangles are as the Sines of their oppoſite Angles.* W. W. D.

PRO-

PROPOSITION XIII.

THEOREM.

In a plane Triangle the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Base, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whose Legs are AB, BC, and Base AC. Produce AB to H, so that BH=BC; then shall AH be the Sum of the Legs; and if you make BI=BA, then IH will be the Difference of the Legs. Also, the Angle HBC=Angles A+ACB (*by* 32 *El.* 1.); and so EBC the Half thereof =half the Sum of the Angles A, and ACB, and its Tangent (putting the Radius=EB) is EC. Again, let BD be drawn paralled to AC, and make HF=CD; then, since HB=CB, we shall have (*by* 4 *El.* 1.) the Angle HBF=CBD=BCA (*by* 29 *El.* 1.) Also, the Angle HBD=Angle A; whence FBD shall be the Difference of the Angles A and ACB, and EBD, whose Tangent is ED, half their Difference. Let IG be drawn thro' I parallel to AC or BD; and then (*by* 2 *El.* 6.) AB : BI : : CD : DG. But AB=BI; whence we shall have CD=DG; but CD=HF, and so HF=DG; and, consequently, HG=DF, and $\frac{1}{2}$ HG=$\frac{1}{2}$ DF=DE; and because the Triangles AHC, IHG, are equiangular, it shall be, as AH : IH : : HC : HG : : $\frac{1}{2}$ HC : $\frac{1}{2}$ HG : : EC : ED. That is, AH *the Sum of the Legs, to* IH *the Difference of the Legs, shall be, as* EC *the Tangent of one half the Sum of the Angles at the Base, to* ED *the Tangent of one half of their Difference.* W. W. D.

PROPOSITION XIV.

THEOREM.

In a plane Triangle, the Base, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Base, are proportional.

LET DC be the Base of the Triangle BCD. About the Centre B, with the Radius BC, let a Circle be described; produce DB to G, and from B let fall BE

perpendicular to the Base; then shall DG=DB+BC =Sum of the Sides, and DH=Difference of the Sides; and DE, CE, are the Segments of the Base whose Difference is DF; then, because (*by Cor. Prop.* 37. *El.* 3.) the Rectangle under DC and DF is equal to the Rectangle under DG, DH, it shall be (*by* 16 *El.* 6.), as D : DG : : DH : DF.

PROPOSITION XV.

THEOREM.

The Sum and Difference of any two Quantities being given, to find the Quantities themselves.

IF one half of the Sum be added to one half of the Difference, the Aggregate shall be equal to the greater of the Quantities; and if from one half of the Sum be taken one half of the Difference, the Residue shall be equal to the lesser of the Quantities. For, let there be two Quantities AB, BC; and let there be taken AD=BC; then DB will be their Difference, and AC their Sum; which, bisected in E, gives AE or EC the half Sum; and DE or EB the half Difference. Hence, AB=AE+EB=the half Sum+the half Difference; and BC=CE—EB=the half Sum —the half Difference.

Note, I. *In any plane Triangle, if two Angles be given the third Angle is also given; because it is their Complement to two Right Angles.*

II. *If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, because it is the Complement of the given Angle to a Right Angle.*

III. *And if two Sides of a Right-angled Triangle be given, the other Side may be found by the first Proposition, without a Canon.*

The Trigonometrical Solutions of a Right-angled Triangle may be as follow. Vid. Fig. A.

	Given	Sought	Make, as
1	The Legs AB and BC	The Angles.	AB : BC : : R : T, of the Angle A, whose Complement is the Angle C.
2	The Leg AB, and the Hypothenuse AC.	The Angles.	AC : AB : : R : S, C, whose Complement is the Angle A.
3	The Leg AB, and the Angle A.	The other Leg BC, and the Hypothenuse AC.	R : T, A : : AB : BC. S, C : R : : AB : AC.
4	The Hypothenuse AC, and the Angle C.	The Leg AB.	R : S, C : : AC : AB.

The Trigonometrical Solutions of Oblique-angled Triangles. Vid. Fig. to Prop. 12

	Given	Sought	Make, as
1	The Angles A, B, C, and the Side AB.	The Sides BC and AC.	S, C : S, A : : AB : BC. Also, S, C : S, B : : AB : AC. But when two Angles are given, the third is also given: Whence the Case wherein two Angles and a Side are given, to find the rest, falls into this Case.

Given

	Given	Sought	Make as
2	All the Angles A, B, C.	All the Sides A B, A C, B C.	S, C : S, A : : AB : BC. And S, C : S, B : AB : AC. Whence, if the Angles are given, the Proportions of the Sides may be found, but not the Sides themselves, unless one of them be first known.
3	The two Sides A B, B C, and C, the Angle opposite to one of them.	The Angles A and B.	AB : BC : : S, C : S, A; which therefore may be found. When AB, the Side opposite to C, the given Angle, is longer than CB, the Side opposite to the sought Angle, the sought Angle is less than a Right one. But when it is shorter, because the Sine of an Angle, and that of its Complement to two Right Angles, is the same, the Species of the Angle A must be first known, or the Solution will be ambiguous.
4	The two Sides A B, B C, and the interjacent Angle B.	The Angles A and C.	*Vid. Fig.* to *Prop.* 13. $BC+AB : BC-AB : : T, \frac{A+C}{2} : T, \frac{A-C}{2}$ Whence is known the Difference of the Angles A and C, whose Sum is given; and so (by *Prop.* 5.) the Angles themselves will be given.
5	All the Sides A B, B C, A C.	The Angles.	*Vid. Fig.* B. Let the Perpendicular be drawn from the Vertex to the Base, and find the Segments of the Base by *Prop.* 14. *viz.* make as BC : AC+AB : : AC—AB : DC—DB. And so BD, DC, are given from this Analogy; and thence the Angles ABD, ACD, will be given by the Resolution of Right-angled Triangles.

THE

ELEMENTS

OF

Spherical Trigonometry.

DEFINITIONS.

I. *THE Poles of a Sphere are two Points in the Superficies of the Sphere, that are the Extremes of the Axis.*

II. *The Pole of a Circle in a Sphere is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle are equal to one another.*

III. *A great Circle in a Sphere is that whose Plane passes thro' the Centre of the Sphere, and whose Centre is the same with that of the Sphere.*

IV. *A spherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

V. *A spherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the said Circles.*

PRO-

PROPOSITION I.

Great Circles ACB, AFB, *mutually bisect each other.*

FOR, since the Circles have the same Centre, their common Section shall be a Diameter of each Circle, and so will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being less than Semicircles, do not comprehend a Space; for they cannot, unless they meet each other in two opposite Points; that is, unless they are Semicircles.

PROPOSITION II.

If from the Pole C *of any Circle* AFB, *be drawn a Right Line* CD *to the Centre thereof, the said Line will be perpendicular to the Plane of that Circle.* Vid. Fig. to Prop. 1.

LET there be drawn any Diameters EF, GH, in the Circle AFB; then, because in the Triangles CDF, CDE, the Sides CD, DF, are equal to the Sides CD, DE, and the Base CF equal to the Base CE (by *Def.* 2.); then (*by* 8 *El.* 1.) shall the Angle CDF=Angle CDE, and so each of them will be a Right Angle. After the same manner we demonstrate, that the Angles CDG, CDH, are Right Angles; and so (by 4 *El.* 11.) CD shall be perpendicular to the Plane of the Circle AFE. W. W. D.

Coroll. 1. A great Circle is distant from its Pole by the Interval of a Quadrant: for, since the Angles CDG, CDF, are Right Angles, the Measures of them, *viz.* the Arcs CG, CF, will be Quadrants.

2. Great Circles, that pass thro' the Pole of some other Circle, make Right Angles with it; and, contrariwise, if great Circles make Right Angles with some other Circle, they shall pass thro' the Poles of that other Circle; for they must necessarily pass thro' the Right Line DC.

PROPOSITION III.

If a great Circle ECF *be described about the Pole* A; *then the Arc* CF, *intercepted between* AC, AF, *is the Measure of the Angle* CAF, *or* CDF. Vid. Fig. to Prop. 1.

THE Arcs AC, AF (*by Cor.* 1. *Prop.* 2.) are Quadrants; and, consequently, the Angles ADC, ADF, are Right Angles: Wherefore (*by Def.* 6. *El.* 11.) the Angle CDF (whose Measure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and also equal to the Spherical Angle CAF, or CBF. W. W. D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, then shall A be the Pole of the Circle passing thro' the Points C and F; for AD is at Right Angles to the Plane FDC (*by* 4 *El.* 11.)

2. The vertical Angles are equal; for each of them is equal to the Inclination of the Circles; also the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles shall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehended by the two Sides also equal.

PROPOSITION V.

Also Triangles shall be equal and congruous, if one Side, together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually equilateral, are also mutually equiangular.

PROPOSITION VII.

In Isosceles Triangles, the Angles at the Base are equal.

PRO-

PROPOSITION VIII.

And if the Angles at the Base be equal, then the Triangle ſhall be Iſoſceles.

THESE five laſt Propoſitions are demonſtrated in the ſame manner, as in plane Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle is the ſhorteſt Way, between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a Spherical Triangle is leſs than a Semicircle.

LET AC, AB, the Sides of the Triangle ABC, be produced till they meet in D: Then ſhall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPOSITION XI.

The three Sides of a Spherical Triangle are leſs than a whole Circle.

FOR BD+DC is greater than BC (*by Prop.* 9.); and, adding on each Side BA+AC; then DBA +DCA, that is a whole Circle, will be greater than BA+BC+AC, which are the three Sides of the Spherical Triangle ABC.

PROPOSITION XII.

In any Spherical Triangle ABC, the greater Angle A is ſubtended by the greater Side.

MAKE the Angle BAD=Angle B; then ſhall AD=BD (*by* 8 *of this*); and ſo BDC=DA+DC, and theſe Arcs are greater than AC. Wherefore the Side BC, that ſubtends the Angle BAC, is greater than the Side AC, that ſubtends the Angle B.

PRO-

PROPOSITION XIII.

In any Spherical Triangle ABC, *if the Sum of the Legs* AB *and* BC *be greater, equal, or less, than a Semicircle, then the internal Angle at the Base* BAC *shall be greater, equal, or less, than the external and opposite Angle* BCD; *and so the Sum of the Angles* A *and* ACB *shall also be greater, equal, or less, than two Right Angles.*

FIRST, let AB+BC=Semicircle=AD; then shall BC=BD, and the Angles BCD and D equal (*by* 8 *of this*); and therefore the Angle BCD shall be= Angle A.

Secondly, let AB+BC be greater than ABD; then shall BC be greater than BD; and so the Angle D (that is, the Angle A, *by* 12 *of this*) shall be greater than the Angle BCD. In like manner we demonstrate, if AB+BC be together less than a Semicircle, that the Angle A will be less than the Angle BCD: And because the Angles BCD and BCA are = two Right Angles, if the Angle A be greater than the Angle BCD, then shall A and BCA be greater than two Right Angles; if the Angle A=BCD, then shall A and BCA be equal to two Right Angles; and if A be less than BCD, then will A and BCA be less than two Right Angles. W. W. D.

PROPOSITION XIV.

In any Spherical Triangle GHD, *the Poles of the Sides, being joined by great Circles, do constitute another Triangle* XMN, *which is the Supplement of the Triangle* GHD; viz. *the Sides* NX, XM, *and* NM, *shall be Supplements of the Arcs that are the Measures of the Angles* D, G, H, *to the Semicircles*; *and the Arcs that are the Measures of the Angles* M, X, N, *will be the Supplements of the Sides* GH, GD, *and* HD, *to Semicircles.*

FROM the Poles G, H, D, let the great Circles XCAM, TMNO, XKBN, be described; then, because

because G is the Pole of the Circle XCAM, we shall have GM=Quadrant (*Cor.* 1. *Prop.* 2.); and since H is the Pole of the Circle TMO, then will HM be also a Quadrant and so (*by Cor.* 1. *Prop.* 3.) M shall be the Pole of the Circle GH. In like manner, because D is the Pole of the Circle XBN, and H the Pole of the Circle TMN, the Arcs DN, HN, will be Quadrants; and so (*by Cor.* 1. *Prop.* 3.) N shall be the Pole of the Circle HD. And because GX, DX, are Quadrants, X will be the Pole of the Circle GD. These Things premised,

Because NK=Quadrant, and XB=Quadrant (*by Cor.* 1. *Prop.* 2.); then will NK+XB, that is, NX +KB=two Quadrants, or a Semicircle; and so NX is the Supplement of the Arc KB, or of the Measure of the Angle HDG, to a Semicircle. In like manner, because MC=Quadrant, and XA=Quadrant, then will MC+XA, that is, XM+AC=two Quadrants, or a Semicircle; and, consequently, XM is the Supplement of the Arc AC, which is the Measure of the Angle HGD. Likewise, since MO, NT, are Quadrants, we shall have MO+NT=OT+NM=Semicircle: And therefore NM is the Supplement of the Arc OT, or of the Measure of the Angle GHD to a Semicircle. W. W. D.

Moreover, because DK, HT, are Quadrants, DK +HT, or KT+HD, are equal to two Quadrants, or a Semicircle; therefore KT, or the Measure of the Angle XNM, in the Supplement of the Side HD to a Semicircle. After the same manner it is demonstrated, that OC, the Measure of the Angle XMN, is the Supplement of the Side GH; and BA, the Measure of the Angle X, is the Supplement of the Side GD. W. W. D.

PROPOSITION XV.

Equiangular Spherical Triangles are also equilateral.

FOR their Supplementals (*by* 14 *of this*) are equilateral, and therefore equiangular also; and so themselves are likewise equilateral (*by Part* 2, *Prop.* 14.)

PRO-

PROPOSITION XVI.

The three Angles of a Spherical Triangle are greater than two Right Angles, and less than six.

FOR the three Measures of the Angles G, H, D, together with the three Sides of the Triangle XNM, make three Semicircles (*by* 14 *of this*); but the three Sides of the Triangle XNM are less than two Semicircles (*by* 11 *of this*). Wherefore the three Measures of the Angles G, H, D, are greater than a Semicircle; and so the Angles G, H, D, are greater than two Right Angles.

The second Part of the Proposition is manifest; for, in every Spherical Triangle, the external and internal Angles, together, only make six Right Angles: Wherefore the internal Angles are less than six Right Angles.

PROPOSITION XVII.

If from the Point R, *not being the Pole of the Circle* AFBE, *there fall the Arcs* RA, RB, RG, RV, *of great Circles to the Circumference of that Circle; then the greatest of those Arcs is* RA, *which passes thro' the Pole* C *thereof; and the Remainder of it is the least; and those that are more distant from the greatest are less than those which are nearer to it, and they make an obtuse Angle with the former Circle* AFB, *on the Side next to the greatest Arc.* Vid. Fig. to Prop. 1.

BEcause C is the Pole of the Circle AFB, then shall CD, and RS, which is parallel thereto, be perpendicular to the Plane AFB. And if SA, SG, SV, be drawn, then shall SA (*by* 7 *El.* 3.) be greater than SG, and SG greater than SV. Whence, in the Right-angled plane Triangles RSA, RSG, RSV, we shall have RSq+SAq, or RAq, greater than RSq+SGq, or RGq; and so RA will be greater than RG, and the Arc RA greater than the Arc RG. In like manner, RSq+SGq, or RGq, shall be greater than RSq+SVq, or

or RVq; and so RG shall be greater than RV, and the Arc RG greater than the Arc RV.

2*dly*, The Angle RGA is greater than the Angle CGA, which is a Right Angle (*by Cor. Prop.* 3.); and the Angle RVA is greater than the Angle CVA, which also is a Right Angle. Therefore the Angles RGA, RVA, are obtuse Angles.

PROPOSITION XVIII.

In Spherical Triangles AGR, AGX, *Right-angled at* A, *the Legs containing the Right Angle are of the same Affection with the opposite Angles; that is, if the Legs be greater or less than Quadrants, then, accordingly, will the Angles opposite to them be greater or less than Right Angles.* Vid. Fig. to Prop. 1.

FOR if AC be a Quadrant, then will C be the Pole of the Circle AEB; and the Angles AGC, AVC, will be Right Angles. If the Leg AR be greater than a Quadrant, then shall the Angle AGR be greater than a Right Angle (*by* 17 *of this*); and if the Leg AX be less than a Quadrant, the Angle AGX shall be less than a Right Angle.

PROPOSITION XIX.

If two Legs of a Right-angled Spherical Triangle be of the same Affection (and consequently the Angles), that is, if they are both less, or both greater, than a Quadrant, then will the Hypothenuse be less than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle ARV, or BRV, let F be the Pole of the Leg AR; then will RF be a Quadrant, which is greater than RV (*by* 17 *of this*).

PROPOSITION XX.

If they be of a different Affection, then shall the Hypothenuse be greater than a Quadrant. Vid. Fig. to Prop. 1.

FOR, in the Triangle ARG, the Hypothenuse RG is greater than RF, which is a Quadrant. PRO-

PROPOSITION XXI.

If the Hypothenuse be greater than a Quadrant, then the Legs of the Right Angle, and so the Angles opposite to them, are of a different Affection; but if lesser, of the same Affection. Vid. Fig. to Prop. 1.

THIS Proposition, being the Converse of the former ones, easily follows from them.

PROPOSITION XXII.

In any Spherical Triangle ABC, *if the Angles at the Base* B *and* C *be of the same Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.*

IN the first Case, if the Perpendicular does not fall within, let it fall without the Triangle (as in *Fig.* 2.); then, in the Triangle ABP, the Side AP is of the same Affection with the Angle B. And, in like manner, in the Triangle ACP, AP is of the same Affection with the Angle ACP; therefore, since ABC and ACP are of the same Affection, the Angles ABC, ACB, shall be of a different Affection; which is contrary to the Hypothesis.

In the second Case, if the Perpendicular does not fall without, let it fall within (as in *Fig.* 1.) Then, in the Triangle ABP, the Angle B is of the same Affection with the Leg AP. So, likewise, in the Triangle ACP, the Angle C is of the same Affection with AP; and therefore the Angles B and C are of the same Affection; which is contrary to the Hypothesis.

PROPOSITION XXIII.

In Spherical Triangles BAC, BHE, *Right-angled at* A *and* H, *if the same acute Angle* B *be at the Base* BA, *or* BH, *then the Sines of the Hypothenuses shall be proportional to the Sines of the perpendicular Arcs.*

FOR the Right Lines CD, EF, being perpendicular to the same Plane, are parallel. Also, FR, DP,

DP, perpendicular to the Radius OB, are likewise parallel: Wherefore the Planes of the Triangles EFR, CDP, are also parallel (*by* 15 *El.* 11.) Wherefore CP, ER, the common Sections of those Planes, with the Plane passing thro' BE, CO, will be parallel (*by* 16 *El.* 11.) Therefore the Triangles CDP, EFR, shall be equiangular: Wherefore CP, the Sine of the Hypothenuse BC, is to CD, the Sine of the perpendicular Arc CA, as ER, the Sine of the Hypothenuse BE, to EF, the Sine of the perpendicular Arc EH. W. W. D.

PROPOSITION XXIV.

The same Things being supposed, AQ, HK, *the Sines of the Bases, are proportional to* IA, GH, *the Tangents of the perpendicular Arcs.*

FOR, after the same manner as in the last Proposition, we demonstrate, that the Triangles QAI, KHG, are equiangular; whence, QA : AI : : KH : HG.

PROPOSITION XXV.

In a Spherical Triangle ABC, *Right-angled at* A, *as the Cosine of the Angle* B, *at the Base* BA, *is to the Sine of the vertical Angle* ACB, *so is the Cosine of the Perpendicular to the Radius.*

PREPARATION.

LET the Sides AB, BC, CA, be produced, so that BE, BF, CI, CH, be Quadrants; and from the Poles B and C draw the great Circles EFDG, IHG; then will the Angles at E, F, I, H, be Right Angles; and so D is the Pole of BAE (*by Cor.* 2. *Prop.* 2. *of this*), and G the Pole of IFCB: Also, AE will be = Complement of the Arc BA, and FE the Measure of the Angle B = GD, and DF their Complement: Also, BC shall be = FI = Measure of the Angle G, and CF their Complement: Likewise, CA = HD, and DC their Complement. These Things premised, in the Triangles HIC, DCF, Right-angled at I and F, and having the same acute Angle C; since BA is less than a Quadrant, it will be, as S, DF:

DF : S, HI : : S, DC : S, HC; that is, the Cofine of the Angle B is to the Sine of the vertical Angle BCA, as the Cofine of CA is to Radius. W. W. D.

PROPOSITION XXVI.

The Cofine of the Bafe : *Cofine of the Hypothenufe* : : R : *Cof. of the Perpendicular.*

FOR in the Triangles AED, CFD, Right-angled at E, F, having the fame acute Angle D; becaufe AE is lefs than a Quadrant, we have, S, EA : S, CF : : S, DA : S, DC. W. W. D.

PROPOSITION XXVII.

S, *of the Bafe* : R : : T, *of the Perpendicular* : T, *of the Angle at the Bafe.*

FOR in the Triangles BAC, BEF, Right-angled at A and E, and having the fame acute Angle B; becaufe AC is lefs than a Quadrant, we have S, BA : S, BE : : T, AC : T, EF. W. W. D.

PROPOSITION XXVIII.

Cof. of the Vertical Angle : R : : T, *of the Perpendicular* : T, *of the Hypothenufe.*

IN the Triangles GIF, GHD, Right-angled at I and H, and having the fame acute Angle G, becaufe HD is lefs than HC, or a Quadrant, it is, as S, GH : S, GI : : T, HD : T, IF.

PROPOSITION XXIX.

S, *of the Hypothenufe* : R : : S, *of the Perpendicular* : S, *of the Angle at the Bafe.*

IN the aforefaid Triangles we have S, IF : S, GF : : S, HD : S, GD.

PRO-

PROPOSITION XXX.

R : *Cof. of the Hypothenufe* : : T, *of the vertical Angle* : *Cot. of the Angle at the Bafe.*

IN the Triangles HIO, DFC, Right-angled at I and F, and having the fame acute Angle C, becaufe DF is lefs than a Quadrant, we have S, CI : S, CF : : T, HI : T, DF; that is, R : Cof. BC : : T, C : Cot. B.

The laft fix Propofitions are fufficient for folving all the fixteen Cafes of Right-angled Spherical Triangles. Thefe fixteen Cafes, with their Analogies deduced from the faid Propofitions, are as follow:

	Given befides the Right Angle	Sought	*Vid. Fig.* to *Prop.* 25, 26, 27, 28, 29, 30.	
1	AC and C.	B.	R : Cof. CA : : S, C : Cof. B, of the fame Kind with CA.	By the Inverfe of Prop. 25.
2	AC and B.	C.	Cof. CA : R : : Cot. B : S, C, this is ambiguous.	By Prop. 25.
3	B and C.	AC.	S, C : Cof. B : : R : Cof. CA, of the fame Kind with the Angle B.	By [illegible] 25, and [illegible].
4	BA, CA.	BC.	R : Cof. BA : : Cof. AC : Cof. BC. If BA, AC, be of the fame Affection, and not Quadrants, then BC will be lefs than a Quadrant. If they be of a different Affection, BC fhall be greater than a Quadrant.	By Prop. 26, [illegible], and 20.

	Given besides the Right Angle	Sought		
5	BC, BA.	AC.	Cos. BA : R :: Cos. BC : Cos. CA. If BC be less than a Quadrant, then shall BA and CA be of the same Affection; if greater, of a different: But BA is given, and therefore the Species thereof. Wherefore the Species of AC is also given.	By Prop. 26, and 21.
6	BA, CA.	B.	S, BA : R :: T, CA : T, B, of the same Affection with the opposite Side CA.	By Prop. 27, and 18.
7	BA, B.	AC.	R : S, BA :: T, B : T, AC, of the same Kind with B.	By Prop. 27, and 18.
8	AC, B.	BA	T, B : T, CA :: R : S, AB, ambiguous.	By Prop. 27.
9	BC, C.	AC.	R : Cos. C :: T, BC : T, CA. If BC be less than a Quadrant, the Angles C and B are of the same Affection; if greater, of a different. Therefore, if the Species of the Angle B be given, then will AC be given.	By Prop. 28, and 21.
10	AC, C	BC.	Cos. C : R :: T, CA : T, BC. And so, if the Angle C and CA, be of the same Affection, then BC shall be lesser than a Quadrant; if of a different, greater.	By Prop. 28, 20, 21.
11	BC, AC.	C.	T, BC : T, CA :: R : Cos. C. If BC be less than a Quadrant, then CA and BA, and consequently the Angles, shall be of the same Affection; if greater, of a different. But the Species of CA is given; therefore the Species of the Angle C will be also given.	By Prop. 28, and 21.

Given

	Given beſides the Right Angle	Sought		
12	BC, B.	AC.	R : S, BC : : S, B : S, AC, of the ſame Species with B.	By Prop. 29, and 18.
13	AC, B.	BC.	S, B : S, AC : : R : S, BC, ambiguous.	By Prop. 29.
14	BC, AC.	B.	S, BC : R : : S, AC : S, B, of the ſame Species with CA.	By Prop. 29.
15	B, C.	BC.	T, C : Cot. B : : R : Coſ. BC. And ſo, if the Angles B and C are of the ſame Affection, then ſhall BC be leſs than a Quadrant; if of a different, greater.	By Prop. 30, 19. and 20.
16	BC, C.	B.	R : Coſ. BC : : T, C : Cot. B. And ſo, if BC be leſſer than a Quadrant, the Angles C and B muſt be of the ſame Affection; if greater, of a different. But the Species of the Angle C is given; therefore the Species of the Angle B will be given alſo.	By Prop. 30, and 31.

Of the Solution of Right-angled Spherical Triangles, by the five circular Parts.

THE Lord *Neper* (the noble Inventor of Logarithms) by a due Consideration of the Analogies by which Right-angled Spherical Triangles are solved, found out two Rules, easy to be remembered, by means of which, all the sixteen Cases may be solved: For since, in these Triangles, besides the Right Angles, there are three Sides, and two Angles; the two Sides comprehending the Right Angle, and the Complements of the Hypothenuse, and the two other Angles, were called by *Neper*, *Circular Parts*; and when there are given any two of the said Parts, and a third is sought; one of these three, which is called the *Middle Part*, either lies between the two other Parts, which are called *Adjacent Extremes*; or is separated from them, and then are called *Opposite Extremes*. So if the Complement of the Angle B (*Fig.* to *Prop.* 25.) be supposed the middle Part, then the Leg AB, and the Complement of the Hypothenuse BC, are adjacent extreme Parts; but the Complement of the Angle C, and the Sides AC, are opposite Extremes. Also, if the Complement of the Hypothenuse BC be supposed the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs AB, AC, are opposite Extremes. In like manner, supposing the Leg AB the middle Part, the Complement of the Angle B and AC are adjacent Extremes; for the Right Angle A doth not interrupt the Adjacence, because it is not a circular Part. But the Complement of the Angle C, and the Complement of the Hypothenuse BC, are opposite Extremes to the said middle Part. These Things premised,

RULE I.

In any Right-angled Spherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

RULE

RULE II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Parts.

Each of the Rules have three Cases: For the middle Part may be the Complement of the Angle B, or C; or the Complement of the Hypothenuse BC; or one of the Legs, AB, AC.

Case 1. Let the Complement of the Angle C be the middle Part; then shall AC, and the Complement of the Hypothenuse BC, be adjacent Extremes. By *Prop.* 28. the Cosine of the vertical Angle C is to Radius as the Tangent of CA is to the Tangent of the Hypothenuse BC. Then (by Alternation) we shall have Cos. C : T, CA : : R : T, BC. But R : T, BC : : Cot. BC : R (as has been before shewn). Wherefore Cos. C : T, AC : : Cot. BC : R; whence R×Cos. C = T, AC×Cot. BC.

And the Complement of the Angle B, and AB, are opposite Extremes to the same middle Part, the Complement of the Angle C; and (*by Prop.* 25.) as the Cosine of the Angle C, to the Sine of the Angle CDF, so is the Cosine of DF to Radius. But the Sine of CDF = AE = Cos. BA, and Cos. DF = S, EF = S, Angle B. Whence it will be, as Cos. C : Cos. BA : : S, B : R. And R×Cos. C=Cos. BA×S, B; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Extremes.

Case 2. Let the Complement of the Hypothenuse BC be the middle Part; then the Complements of the Angles B and C will be adjacent Extremes. In the Triangle DCF (*by Prop.* 27.) it is, as S, CF : R : : T, DF : T, C. Whence (by Alternation) S, CF : T, DF : : (R : T, C : :) Cot. C : R. But S, CF = Cos. BC and T, DF=Cot. B. Wherefore R×Cos. BC= Cot. C×Cot. B; that is, Radius drawn into the Sine of the middle Part is equal to the Product of the Tangents of the adjacent extreme Parts.

 And

And BA, AC, are the opposite Extremes to the said middle Part, *viz* the Complement of BC; and (*by Prop.* 26.) Cof. BA : Cof. BC : : R : Cof. AC. Wherefore we shall have R × Cof. BC = Cof. BA × Cof. AC.

Case 3. Lastly, let AB be the middle Part; and then the Complement of the Angle B, and AC, will be adjacent Extremes, and (*by Prop.* 27.) S, AB : R : : T, CA : T, B. Whence S, AB : T, CA : : (R : T, B : :) Cot. B : R. And so R × S, AB = T, CA = Cot. B.

Moreover, the Complements of BC, and the Angle C, are opposite Extremes to the same middle Part AB; and in the Triangle GHD (*by Prop.* 25.) we have Cof. D : S, DGH : : Cof. GH : R. But Cof. D = Cof. AE = S, AB. and S, G = S, IF = S, BC. Also, Cof. GH = S, HI = S, C. Wherefore it will be, as S, AB : S, BC : : S, C : R. And hence R × S, AB = S, BC × S, C.

And so, in every Case, the Rectangle under the Radius, and the Sine of the middle Part, shall be equal to the Rectangle under the Cosines of the opposite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes: And, consequently, if the aforesaid Equations be resolved into Analogies (*by* 16 *El.* 6.), the unknown Parts may be found by the Rule of Proportion. And if that Part sought be the middle one, then shall the first Term of the Analogy be Radius, and the second and third, the Tangents or Cosines of the extreme Parts. If one of the Extremes be sought, the Analogy must begin with the other; and the Radius, and the Sine of the middle Part, must be put in the middle Places, that so the Part sought may be in the fourth Place.

IN Oblique-angled Spherical Triangles (*Fig. to Prop.* 31.) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Base, continued, if need be, so as to make two Right-angled Spherical Triangles BAC, DAC; then, by those Right-angled Triangles, may most of the Cases of Oblique-angled ones be solved.

PRO-

PROPOSITION XXXI.

The Cosines of the Angles B *and* D, *at the Base* BD, *are proportional to the Sines of the vertical Angles* BCA, DCA.

FOR Cos. Angle B : S, BCA : : (Cos. CA : R : :) Cos. D : S, DCA (*by* 25 *of this.*)

PROPOSITION XXXII.

The Cosines of the Sides BC, DC, *are proportional to the Cosines of the Bases* BA, DA.

FOR Cos. BC : Cos. BA : : (Cos. CA : R : :) Cos. DC : Cos. DA (*by* 26 *of this.*)

PROPOSITION XXXIII.

The Sines of the Bases BA, DA, *are in a reciprocal Proportion of the Tangents of the Angles* B *and* D, *at the Base* BD.

BEcause (*by* 27 *of this*) S, BA : R : : T, AC : T, of the Angle B. And by the same inversly, R : S, DD : : T, of the Angle D : T, AC. Then will it be (by the Equality of perturbate Ratio, according to *Prop.* 23. *El.* 5.) S, BA : S, DA : : T, Angle D : T, Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides BC, DC, *are in a reciprocal Proportion of the Cosines of the vertical Angles* BCA, DCA.

BEcause, by alternating the 28th *Proposition*; we have T, BC : R : : T, CA : Cos. BCA, and by the same, R : Cos. DCA : : T, DC : T, CA. Wherefore, by Equality of perturbate Proportion, T, BC : Cos. DCA : : T, DC : Cos. BCA.

PROPOSITION XXXV.

The Sines of the Sides BC, DC, *are proportional to the Sines of the opposite Angles* D *and* B.

BEcause (*by the 29th of this*) S, BC . R : : S, CA : S, of the Angle B; and, by the same, inverting, R : S, DC : : S, Angle D : S, of CA. Whence, by Equality of perturbate Ratio, S, BC : S, DC : : S, D : S, B.

PROPOSITION XXXVI.

In any Spherical Triangle ABC, *the Rectangle* CF × AE, *or* FM × AE, *contained under the Sines of the Legs* BC, BA, *is to the Square of the Radius, as* IL *or* IA—LA *the Difference of the versed Sines of the Base* CA, *and the Difference of the Legs* AM, *to* GN, *to the versed Sine of the Angle* B.

LET a great Circle PN be described from the Pole B; and let BP, BN, be Quadrants; and then PN is the Measure of the Angle B: also, describe from the same Pole B a lesser Circle CFM thro' C; the Planes of these Circles shall be perpendicular to the Plane BON (*by the 2d of this.*) And PG, CH, being perpendicular in the same Plane, fall on the common Sections ON, FM; suppose in G, H. Again, draw HI perpendicular to AO; and then the Plane drawn thro' CH, HI, shall be perpendicular to the Plane AOB. Whence AI, which is perpendicular to HI, will be perpendicular to the Right Line CI (*by Def. 4. El. 11.*); and so AI is the versed Sine of the Arc AC, and AL the versed Sine of the Arc AM = BM—BA = BC—BA. The Isosceles Triangles CFM, PON, are equiangular, since MF, NO, as also CF, PO (*by 16 El. 11.*) are parallel. Wherefore, if Perpendiculars CH, PG, be drawn to the Bases FM, ON, the Triangles will be divided similarly, and we shall have FM : ON : : MH : GN. Also, because the Triangles AOE, DIH, DLM, are equiangular, we shall have AE : AO : : IL : MH.

But

But it has been proved, that FM : ON : : MH : GN. Wherefore it ſhall be, as AE×FM : AO×ON : : IL×MH : MH×GN, or ſo is IL to GN; that is, the Rectangle under the Sines of the Legs, is to the Square of Radius, as the Difference of the verſed Sines of the Baſe, and the Difference of the Legs BC, BA, is to the verſed Sine of the Angle B. W. W. D.

PROPOSITION XXXVII.

The Difference of the verſed Sines of two Arcs, drawn into half the Radius, is equal to the Rectangle under the Sine of half the Sum, and the Sine of half the Difference, of thoſe Arcs.

LET there be two Arcs, BE, BF, whoſe Difference EF let be biſected in D; then ſhall BD be the half Sum, and FD the half Difference of thoſe Arcs. GE =IL is the Difference of the verſed Sines of the Arcs BE, BF; alſo, FO is the Sine of the half Difference of the Arcs. And becauſe the Triangles CDK, FEG, are equiangular, we have DK : GE : : (CD : FE : :) $\frac{1}{2}$CD : $\frac{1}{2}$FE. Whence DK×$\frac{1}{2}$ FE, or DK×FO= GE×$\frac{1}{2}$CD=IL×$\frac{1}{2}$CD. W. W. D.

PROPOSITION XXXVIII.

The verſed Sine of any Arc, drawn into half the Radius, is equal to the Square of the Sine of one half of the ſaid Arc.

THE Triangles CBM, DEB, are equiangular, ſince the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore EB : BD : : BM, : BC. And then will EB×BC=BM×BD; and EB×$\frac{1}{2}$ BC=BM×$\frac{1}{2}$ BD=BMq. W. W. D.

PROPOSITION XXXIX.

In any Spherical Triangle ABC, *whose Legs, containing the Angle* B, *are* BC, AB, *and Base subtending that Angle* AC; *if the Arc* AM *be taken = Difference of the Legs* = BC—AB; *then shall the Rectangle under the Sines of the Legs* BC, BA, *be to the Square of the Radius, as the Rectangle, under the Sine of the Arc* $\frac{AC+AM}{2}$ *and the Sine of the Arc* $\frac{AC-AM}{2}$, *is to the Square of the Sine of one half of the Angle* B. Vid. Fig. to Prop. 36.

BEcause the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the versed Sine of the Angle B, or as $\frac{1}{2}$ R×IL to $\frac{1}{2}$ R drawn into the versed Sine of the Angle B (*by Prop.* 36. *of this.*) And since $\frac{1}{2}$ R×IL=Rectangle, under the Sines of the Arcs $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$ (*by Prop.* 37. *of this.*) And also $\frac{1}{2}$ R drawn into the versed Sine of the Angle B is equal to the Square of the Sine of one half of the Angle B (*by Prop.* 38. *of this.*) Therefore the Rectangle under the Sines of the Sides, to the Square of Radius, shall be as the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$ is to the Square of the Sine of one half the Angle B. W. W. D.

The

The twelve Cases of oblique-angled Spherical Triangles are as follow:

	Given	Sought	Make, as	
1	Angles B, D, and BC.	Angle C.	R : Cos. BC : : T, B : Cot. BCA (by Prop. 30. of this): Also Cot. B : S, DCA : : Cot. D : S, DCA (by 31. of this). Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = BCD. Whether the Perpendicular falls within, or without the Triangle, may be known from the Affection of the Angles B and D (by 22. of this); which Admonition ought to be observed in the following Solutions.	*In the Original the Proportion was thus;* Cos. BC : R : : T, B : Cot. BCA.
2	Angles B, BCD, and the Side BC.	Angle D.	R : Cos. BC : : T, B : Cot. BCA (Prop. 30. of this.) And S, BCA : S, DCA : : Cos. B : Cos. D (by Prop. 31.) If BCA be less than BCD, the Angle D shall be of the fame Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D shall be of a different Affection, by the Converse of *Prop.* 22.	*This Proportion in the Original was as in the foregoing. The Species of the Angle* BCA *may be known by* Prop. 18, *and* 19.
3	The Sides BC, CD, and the Angle B.	The Side BD.	R : Cos. B : : T, BC : T, BA (by 28. of this.) And Cos. BC : Cos. BA : : Cos DC : Cos. DA (by 32. of this.) The Sum or Difference of BA and DA, according as the Perpendicular falls within or without the Triangle, is equal to BD; which cannot be known, unless the Species of the Angle D be first known.	

Given

	Given	Sought	Make, as
4	The Sides BC, DB, and the Angle B.	The Side CD.	. : Cos. B : : T, BC : T, BA (by 28. of this.) And Cos. BA : Cos. BC : : Cos. DA : Cos. DC (by Prop. 32. of this) According as DA is similar or dissimilar to CA, or to the Angle BDC, so shall DC be lesser or greater than a Quadrant (by 19, and 20. of this.)
5	Angles B, D. and the Side BC	The Side BD.	: : Cos. B : : T, BC : T, BA (by 28. of this.) And T, D : T, B : : S, BA : S, DA (by 33. of this.) The Sum or Difference of BA and DA = BD.
6	The Sides BC. BD, and the Angle B	Angle D.	Cos. B : : T, BC : T, BA (by Prop. 28. of this.) And S, DA : S, BA : : T, B . T, D (by 33. of this.) According as BD is greater or lesser than BA, the Angle D shall be similar or dissimilar to the Angle B (by 22. of this.)
7	The Sides BC, DC, and the Angle B.	Angle C.	Cos. BC : R : : Cot. B : T, BCA (by 30. of this.) And T, DC : T, BC : : Cos. BCA : Cos. DCA (by 34. of this.) The Sum or Difference of the Angles BCA, DCA, according as the Perpendicular falls within or without the Triangle, is equal to the Angle BCD.
8	The Angles BCD, and B and the Side BC.	The Side DC.	Cos. BC : R : : Cot. B : T, BCA (by 30. of this.) Also, Cos. DCA : Cos. BCA : : T, BC : T, DC (by 34. of this.) If the Angle DCA be similar to the Angle B (that is, if AD be similar to CA), then DC shall be less than a Quadrant. If the Angles DCA and B be dissimilar, then DC shall be greater than a Quadrant, which follows (from Prop. 18, 19, and 20. of this.)

	Given	Sought	Make, as
9	The Sides BC, DC, and the Angle B.	The Angle D.	S, CD : S, B : : S, BC : S, D; which is ambiguous. The Analogy follows from *Prop.* 35. of this.
10	The Angles B, D, and the Side BC.	The Side DC.	S, D : S, BC : : S, B : S, DC; which Side is ambiguous.
11	All the Sides AB, BC, CA. *Vid. Fig. Prop.* 36.	The Angle B.	As the Rectangle under the Sines of the Legs AB, BC : the Square of Radius : : the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B (*by Pro.* 39.)
12	All the Angles G, H, D. *Vid. Fig. Prop.* 14.	The Side GD.	In the Triangle XNM, the Arc MN is the Complement of the Angle GHD to a Semicircle. XM is the Complement of the Angle G, and XN the Complement of the Angle D : And the Angle X, the Complement of the Side GD to a Semicircle. Wherefore, if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the fame as in *Cafe* 11. of this; fince Arcs, and their Complements to Semicircles, have the fame Sines.

The following REMARK, SAMUEL CUNN.

THAT this is true but in a particular Case, *viz.* when two of the Angles of the Triangle are Right ones, and two of the Sides Quadrants, may be thus demonstrated: For, if possible, let some Triangle RST, *Fig.* to *Prop.* 14th, be such, that its Sides RS, ST, TR, be equal to the Measures of GHD, HGD, GDH, the Angles of a Triangle GHD; and, also, that the Measures of RST, STR, TRS, the Angles of the Triangle RST, be equal to GH, GD, HD, the Sides of the Triangle GHD; and produce MX, MN, two Sides of the supplemental Triangle, to Semicircles, and they will meet somewhere, suppose at E; and there will be constructed thereby the Triangle NEX, of which XE (the Supplement of XM, which, by the 14th *Prop.* was the Supplement of the Measure of the Angle HGD) is equal to the Measure itself of the same Angle HGD: And, in like manner, NE, the Supplement of NM, which, by the 14th *Prop.* was the Supplement of the Measure of the Angle GHD, is equal to the Measure itself of the same Angle GHD. But the third Side XN is not the Measure of the third Angle GDH, but its Supplement, by the 14th *Prop.* Moreover, of the Angle EXN (whose Supplement is NXM), the Measure, by the 14th *Prop.* is equal to GD; and of the Angle XNE (whose Supplement is MNX) the Measure, by the 14th *Prop.* is equal to HD. But of the third NEX (which is equal to NMX) the Measure is not equal to GH, but its Supplement.

Now make NV=RT=BK, the Measure of the Angle GDH, and draw the great Circle EV. And since RS, by Supposition, is equal to the Measure of the Angle GHD, which is equal to EN; and since the Measure of the Angle SRT is, by Supposition, equal to DH, which is also equal to the Measure of the Angle XNE; the Angle XNE is equal to the Angle R. Then, consequently, by the 4th *Prop.* the Triangles SRT, ENV, will have the Base ST equal to the Base EV;

EV, the Angle T to the Angle NVE; and the Angle S to the Angle NEV. But ST (which is equal to EV) by Supposition, is equal to the Measure of the Angle HGD; to which Measure XE is also equal: Therefore EV is equal to XE; and, consequently, by the 7th *Prop.* the Angle EVX is equal to the Angle EXV; and the Angle EXV (whose Measure, as hath been shewn above, is equal to GD) is equal to the Angle T (or NVE), since, by Supposition, the Measure of this is also equal to GD. Therefore the Angle EVX is equal to the Angle EVN, and so both Right ones; and, consequently, EXV a Right one also. Therefore, by the 2d *Cor.* to the 2d *Prop.* EV and EX are both Quadrants.

But if EV be a Quadrant, and at Right Angles to NX, then E, by 2d *Prop.* and its *Coroll.* is the Pole of NX; and so EN a Quadrant also, and the Angle ENV a Right one. Therefore, if the Sides of a Triangle (NEV, or its Equal) RST are equal to the Measures of the Angles of some other Triangle GHD, and the Measures of the Angles of the former, equal to the Sides of the latter; two Sides of such a Triangle RST, or GHD, must be Quadrants, and two Angles of each Right ones.

Therefore, if a Triangle RST be constructed, whose Sides are equal to the Measures of the Angles of another Triangle GHD; the Measure of the Angles of the Triangle RST shall not be equal to the Sides of the Triangle GHD, unless in the one Case before-mentioned. Therefore the Measures of the Angles of the Triangle GHD, used as the Sides of a Triangle in the 11th *Case*, will not give us a Side of GHD, but the Measure of an Angle of the Triangle RST, unless in the one afore-mentioned Case; *which was to be demonstrated.*

But to find a Side GD of a Spherical Triangle GHD, whose Angles are all given, produce MN, that Side of the Supplement Triangle, which is equal to the Supplement of the Measure of GHD, the Angle opposite to the Side sought, and MX, either of the other Sides, till they meet, as in E. And there, as hath been before shewn, the Sides EX, EN, of the Triangle EXN,

are exactly equal to the Measures of the Angles HG GHD, of the Triangle GHD; and of the Angles ENX, of the Triangle EXN, the Measure to GE, HD. But the Side XN is equal plement of the Measure of the Angle GDH the Angle XEN, the Measure is equal to ment of GH.

Therefore the SOLUTION is thus

Change one of the Angles GDH, adjacent to the Side fought, into its Supplement; and then work with the Measures of the Angles as tho' they were Sides; and the Result will be GD, the Side fought.

The preceding Fault, as well as the Omissions hereafter mentioned, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caswell*, Mr. *Heynes*, and many other Trigonometrical Writers.

In the Solution of our 9th and 10th Cases, they have told us, that the *Quæsita* are ambiguous; which sometimes, indeed, is true, but sometimes also false. Therefore, as I conceive it, they ought to have laid down Rules, by Help of which we might discover when the *Quæsita* are ambiguous, and when not.

This Overfight may be corrected by the following Directions; wherein, because every Sine corresponds to two Arcs, to one less than a Quadrant, and to another, which is the Supplement of the former to a Semicircle (a true Distinction of which of these are to be used, being necessary to be known, before a proper Solution can be given to such Problems as these are), I shall beg Leave, for Brevity-sake, to call the lesser Arc the acute Value, and the greater the obtuse; whether the Sine be of an Angle, or a Side.

In the tenth Case, there are given two Angles, B, *and* D, *and* BC, *a Side opposite to one of those Angles* D, *to find* DC, *the Side opposite to the other.*

TO the acute Value of DC, and also to its obtuse one, add BC; and if each of these Sums are

$\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than a Semicircle, when the Sum of the Angles B, D, is $\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than two Right Angles; both the Values of DC may be admitted, and then it is ambiguous: But when only one of thoſe Sums is $\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than a Semicircle, only one Value of DC can be true, *viz.* the $\begin{Bmatrix}\text{obtuſe}\\ \text{acute}\end{Bmatrix}$ one; and then it is not ambiguous.

In the ninth Caſe, there are given two Sides BC, DC, *and one Angle* B, *oppoſite to* DC, *one of thoſe Sides, to find* D *the Angle oppoſite to the other.*

TO the acute Value of D, and alſo to its obtuſe Value, add B; and if each of theſe Sums is $\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than two Right Angles, when the Sum of the Sides is $\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than a Semicircle, both the Values of D may be admitted, and conſequently D is ambiguous: But when only one of thoſe Sums is $\begin{Bmatrix}\text{greater}\\ \text{leſs}\end{Bmatrix}$ than two Right Angles, only one Value of D is true, *viz.* the $\begin{Bmatrix}\text{obtuſe}\\ \text{acute}\end{Bmatrix}$ one; and then it is not ambiguous.

Nor are we better uſed in the firſt Caſe; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles; yet, in each of thoſe Varieties, the *Quæſita* will be ſometimes ambiguous, and ſometimes not.

In

In the first Case there are given two Angles B, D, and BC, *a Side opposite to* D, *one of them, to find* C *the third Angle.*

Let the Perpendicular fall within; that is, let the given Angles be of the same Species.

TO the acute Value of DCA, and also to its obtuse one, add the Angle BCA; and if each of the Sums is less than two Right Angles, then either the acute Value of DCA, or its obtuse one added to BCA, gives a Value of BCD; which, therefore, is ambiguous. And when only one of these Sums is less than two Right Angles, the acute Value of DCA, added to BCA, gives the only Value of BCD; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of a different Species.

WHEN the obtuse Value of the Angle DCA is less than the Angle BCA, the Angle BCD may be had by subtracting either Value of DCA from BCA; and then BCD is ambiguous. But when the obtuse Value of DCA is not less than BCA, the acute Value of DCA, taken from BCA, gives the single Value of BCD; which, therefore, is not ambiguous; tho' in both Varieties the Perpendicular fell without.

In the fifth Case we lie under the same Misfortune, where there are given, as in the first, the Angles B, D, *and the Side* BC, *to find* BD *the Side lying between those given Angles.*

1. When the Perpendicular falls within; that is, when the given Angles are of the same Species.

TO the acute Value of DA, and so also to its obtuse one, add BA; and if each of these Sums is less than a Semicircle, then either the acute Value of DA, or its obtuse one, added to BA, gives the Value of BD; which

which thence is ambiguous. And when only one of these Sums is lefs than a Semicircle, the acute Value of DA, added to BA, gives the only Value of BD; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the obtufe Value of DA is lefs than BA, BD will be had by fubtracting either Value of DA from BA; and then BD is ambiguous. But when the obtufe Value of DA is not lefs than BA, the acute Value of DA, taken from BA, leaves the only Value of BD; which, therefore, is not ambiguous, tho' in both Varieties the Perpendicular fell without.

In the third, we have the fame Omiffion; where there are given two Sides BC, CD, *and* B *an Angle oppofite to* CD *one of them, to find the third Side* BD.

FIRST, we may obferve, that the Species of DA is always known; for it is of $\left\{ \begin{array}{c} \text{the fame} \\ \text{a different} \end{array} \right\}$ Affection with the Angle B, when DC is $\left\{ \begin{array}{c} \text{lefs} \\ \text{greater} \end{array} \right\}$ than a Quadrant. And,

If AD be lefs than AB, and alfo the Sum of AD and AB lefs than a Semicircle; then AD, either added to, or fubtracted from AB, will give the Value of BD; which, therefore, is ambiguous.

But if AD be not lefs than AB, or if their Sum be not lefs than a Semicircle; then their Sum in the former, and their Difference in the latter Variety, fhall give one fingle Value of BD; and then it is not ambiguous.

The seventh Case much resembles the third; for there are given two Sides BC, CD, *and* B, *an Angle, opposite to* CD *one of them; to find the Angle* BCD, *lying between those two Sides.*

AND here we may observe, that the Species of the Angle DCA is known; for it is of $\left\{\begin{array}{l}\text{the same}\\\text{a different}\end{array}\right\}$ Kind with the Angle B, when DC is $\left\{\begin{array}{l}\text{less}\\\text{greater}\end{array}\right\}$ than a Quadrant. And,

If DCA be less than BCA, and the Sum of DCA and BCA less than two Right Angles; then DCA, either added to, or subtracted from, BCA, will give the Angle BCD; which, therefore, is ambiguous.

If DCA be not less than BCA, or the Sum of DCA and BCA not less than two Right Angles; then their Sum in the former, and their Difference in the latter, Variety, shall give the single Value of BCD; which, then, is not ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side CD, or the Angle D, as taught in the Remarks on the 9th and 10th Cases; they will find the several Varieties in the 1st, 3d, 5th, and 7th, to be as here laid down in these easy Rules.

The Truth of these Rules may be easily deduced from the 10th, 13th, 18th, and 22d *Prop. of this*; and the 2d, 8th, and 13th *Examples*, following *Prop.* 30. *of this.*

In our third *Case* of oblique plane Triangles, our Author should have added this:

If AB be less than BC, the Angle A is ambiguous; otherwise, not.

A SHORT TREATISE OF THE Nature *and* Arithmetic OF *LOGARITHMS.*

The PREFACE.

THE Mathematics formerly received considerable Advantages; first, by the Introduction of the Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped, at least, as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematics. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for*

other Phænomena of Nature; and, lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom, taking in the whole Doctrine, have indeed written learnedly, but scarcely intelligibly to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature, and more intimate Properties, untouched. My Design therefore in the following Tract is, to supply what seemed still wanting, viz. *to discover and explain the Doctrine of Logarithms, to those whose are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the first that constructed and published a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The same Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr.* Henry Briggs, *Professor of* Geometry *at* Oxford, *who was hereby introduced as a Sharer in the compleating thereof: But the Lord* Neper *dying, the whole Business remaining was devolved upon Mr.* Briggs, *who with prodigious Application and an uncommon*

Dex-

Dexterity, composed a Logarithmic Canon, agreeable to that new Form, for the first twenty Chiliads of Numbers (or from 1 *to* 20000*), and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Gouda *in* Holland, *in the Year* 1628, *with the intermediate Chiliads, before omitted, filled up according to* Briggs's *Prescriptions; but these Tables are not so useful as* Briggs's, *because the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *has also calculated the Logarithms of the Sines and Tangents of every Degree, and the Hundredth Parts of Degrees, to* 15 *Places of Figures; and has subjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called artificial Sines and Tangents. These Tables, together with their Construction and Use, were publish'd after* Briggs's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called* Trigonometria Britannica.

Since then, there have been publish'd, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be sufficient for most Uses.*

The best Disposition of these Tables, in my Opinion, is that first thought of by Nathanael Roe, *of* Suffolk; *and, with some Alterations for the*

 better,

better, followed by Sherwin *in his Mathematical Tables, published at* London *in* 1705, *wherein are the Logarithms from* 1 *to* 101000, *consisting of seven Places of Figures. To which are subjoined the Differences, and proportional Parts, by means of which, may be found easily the Logarithms of Numbers to* 10000000; *observing, at the same Time, that these Logarithms consist only of seven Places of Figures. Here are also the Sines, Tangents, and Secants, with their Logarithms, and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematics.*

OF THE

Nature *and* Arithmetic

OF

LOGARITHMS.

CHAP. I

Of the ORIGIN *and* NATURE *of* *LOGARITHMS.*

AS in Geometry the Magnitudes of Lines are often defined by Numbers; so, likewise, on the other Hand, it is sometimes expedient to expound Numbers by Lines, *viz.* by assuming some Line which may represent Unity; the Double thereof, the Number 2; the Triple, 3; the one Half, the Fraction $\frac{1}{2}$; and so on. And thus the Genesis and Properties of some certain Numbers are better conceived, and more clearly considered, than can be done by abstract Numbers.

Hence, if any Line a* be drawn into itself, the Quantity a^2, produced thereby, is not to be taken as one of two Dimensions, or as a Geometrical Square, whose Side is the Line a, but as a Line that is a third Proportional to some Line taken for Unity, and the Line a. So, likewise, if a^2 be multiplied by a, the Product a^3 will not be a Quantity of three Dimensions, or a Geometrical Cube, but a Line that is the fourth Term in a Geometrical Progression, whose first Term is 1, and second a; for the Terms 1, a, a^2, a^3, a^4, a^5, a^6, a^7, &c. are in the continual Ratio of 1 to a. And the Indices affixed to the Terms shew the Place or Distance that every Term is from Unity. For Example,

* Fig. 1.

ample, a^5 is in the fifth Place from Unity, a^6 in the sixth, or six Times more distant from Unity than a, or a^1, which immediately follows Unity.

If, between the Terms 1 and a, there be put a mean Proportional, which is $\surd a$, the Index of this will be $\frac{1}{2}$, for its Distance from Unity will be one half of the Distance of a from Unity; and so $a^{\frac{1}{2}}$ may be written for $\surd a$. And if a mean Proportional be put between a and a^2, the Index thereof will be $1\frac{1}{2}$, or $\frac{3}{2}$; for its Distance will be sesquialteral of the Distance of a from Unity.

If there be two mean Proportionals put between 1 and a; the first of them is the Cube Root of a, whose Index must be $\frac{1}{3}$; for that Term is distant from Unity only by a third Part of the Distance of a from Unity; and so the Cube Root may be expressed by $a^{\frac{1}{3}}$. Hence the Index of Unity is 0; for Unity is not distant from itself.

The same Series of Quantities, geometrically proportional, may be both Ways continued, as well descending towards the Left Hand, as ascending towards the Right; for the Terms $\frac{1}{a^5}$, $\frac{1}{a^4}$, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, a^5, &c. are all in the same Geometrical Progression. And since the Distance of a from Unity is towards the Right Hand, and positive or $+1$, the Distance equal to that on the contrary Side, *viz.* the Distance of the Term $\frac{1}{a}$, will be negative, or -1, which shall be the Index of the Term $\frac{1}{a}$, for which may be written a^{-1}. So likewise in the Terms a^{-2}, the Index -2 shews, that the Term stands in the second Place from Unity towards the Left Hand, and the Expressions a^{-2} and $\frac{1}{a^2}$ are of the same Value. Also a^{-3} is the same as $\frac{1}{a^3}$. For these negative Indices shew, that the Terms belonging to them go from Unity the contrary

contrary Way to that by which the Terms, whose Indices are positive, do. These Things premised,

If on the Line AN, both Ways indefinitely extended, be taken AC, CE, EG, GI, IL, on the Right Hand; and also AΓ, ΓΠ, &c. on the Left; all equal to one another; and if, at the Points Π, Γ, A, C, E, G, I, L, be raised to the Right Line AN, the Perpendiculars ΠΣ, ΓΔ, AB, CD, EF, GH, IK, LM, which let be continually proportional, and represent Numbers, whereof AB is Unity. The Lines AC, AE, AG, AI, AL,—AΓ,—AΠ, respectively express the Distances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical Proportionals, according as it is distant from Unity. So since AG is triple of the Right Line AC, the Number GH shall be in the third Place from Unity, if CD be in the first: So likewise shall LM be in the fifth Place since AL=5AC. If the Extremities of the Proportionals, Σ, Δ, B, D, F, H, K, M, be joined by Right Lines, the Figure ΣΠ LM will become a Polygon consisting of more or less Sides, according as there are more or less Terms in the Progression.

If the Parts AC, CE, EG, GI, IL, be bisected in the Points *c*, *e*, *g*, *i*, *l*, and there be again raised the Perpendiculars *c d*, *e f*, *g h*, *i k*, *l m*, which are mean Proportionals between AB, CD; CD, EF; EF, GH; GH, IK; IK, LM; then there will arise a new Series of Proportionals, whose Terms, beginning from that which immediately follows Unity, are double of those in the first Series, and the Differences of the Terms are become less, and approach nearer to a Ratio of Equality than before. Likewise in this new Series, the Right Lines AL, AC, express the Distances of the Terms LM, CD, from Unity; *viz.* since AL is ten Times greater than A*c*, LM shall be the tenth Term of the Series from Unity: And because A *e* is three Times greater than A*c*, *ef* will be the third Term of the Series, if *c d* be the first; and there shall be two mean Proportionals between AB and *ef*; and between AB and LM there will be nine mean Proportionals.

And if the Extremities of the said Lines, *viz.* B, *d*, D, *f*, F, *h*, H, &c. be joined by Right Lines, there will be a new Polygon made, consisting of more, but shorter Sides than the last.

If,

If, again, the Distances *Ac*, *c*C, C*e*, *e*L, &c. be supposed to be bisected, and mean Proportionals between every two of the Terms be conceived to be put at their middle Distances; then there will arise another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Difference of the Terms will be less; and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be lesser, because of the Diminution of the Distances of the Terms from each other.

Now, in this new Series, the Distances AL, AC, &c. will determine the Orders or Places of the Terms; *viz.* if AL be five Times greater than AC, and CD be the fourth Term of the Series from Unity, then LM will be the twentieth Term from Unity.

If in this manner mean Proportionals be continually placed between every two Terms, the Number of Terms at last will be made so great, as also the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon so lessened, as to become less than any given Right Line; and consequently the Polygon will be changed into a curve-lined Figure; for any curve-lined Figure may be conceived as a Polygon, whose Sides are infinitely small, and infinite in Number.

A Curve described after this manner is called *Logarithmical*; in which, if Numbers be represented by Right Lines standing at Right Angles to the Axis AN, the Portion of the Axis intercepted between any Number and Unity shews the Place or Order, which that Number obtains in the Series of Geometrical Proportionals, distant from each other by equal Intervals. For Example; if AL be five Times greater than AC, and there are a thousand Terms in continual Proportion, from Unity to LM; then will there be two hundred Terms of the same Series from Unity to CD, or CD shall be the two hundredth Term of the Series from Unity; and let the Number of Terms from AB to LM be supposed what it will, then the Number of Terms from AB to CD will be one fifth Part of that Number.

The

The Logarithmical Curve may alſo be conceived to be deſcribed by two Motions, one of which is equable, and the other accelerated, or retarded, according to a given Ratio. For Example, if the Right Line AB moves uniformly along the Line AN, ſo that the End thereof deſcribes equal Spaces in equal Times; and, in the mean Time, the ſaid Line AB ſo increaſes, that the Increments thereof, generated in equal Times, be proportional to the whole increaſing Line, that is, if AB, in going forward to *cd*, be increaſed by the Increment *od*, and in an equal Time when it is come to CD, the Increment thereof is D*p*, and D*p* to *dc* is as *do* is to AB; that is, if the Increments generated in equal Times are always proportional to the Wholes; or, if the Line AB, moving the contrary Way, diminiſhes in a conſtant Ratio, ſo that while it goes thro' the equal Spaces, the Decrements AB — ΓΔ, ΓΔ — ΠΣ, are proportional to AB, ΓΔ; then the End of the Line, increaſing or decreaſing in the ſaid manner, deſcribes the Logarithmical Curve. For ſince AB : *do* . : *dc* : D*p* : : DC . *fq*; it ſhall be (by Compoſition of Ratio), as AB : *dc* : : *dc* : DC : : DC : *fe*, and ſo on.

By theſe two Motions, *viz.* the one equable, and the other proportionally accelerated or retarded, the Lord *Neper* laid down the Origin of Logarithms, and called the Logarithm of the Sine of any Arc, *That Number which neareſt defines a Line that equally increaſes, while, in the mean Time, the Line expreſſing the whole Sine proportionally decreaſes to that Sine.*

It is manifeſt, from this Deſcription of the Logarithmic Curve, that all Numbers at equal Diſtances are continually proportional. It is alſo plain, that if there be four Numbers AB, CD, IK, LM, ſuch, that the Diſtance between the firſt and ſecond be equal to the Diſtance between the third and the fourth: Let the Diſtance from the ſecond to the third be what it will, theſe Numbers will be proportional. For, becauſe the Diſtances AC, IL, are equal, AB ſhall be to the Increment D*s*, as IK is to the Increment MT. Wherefore (by Compoſition) AB : DC : IK : ML. And contrariwſe, if four Numbers be proportional, the Diſtance between the firſt and the ſecond ſhall

be

be equal to the Diſtance between the third and fourth.

The Diſtance between any two Numbers is called the Logarithm of the Ratio of thoſe Numbers, and indeed doth not meaſure the Ratio itſelf, but the Number of Terms in a given Series of Geometrical Proportionals proceeding from one Number to another; and defines the Number of equal Ratios by the Compoſition whereof the Ratios of Numbers are known.

If the Diſtance between any two Numbers be double to the Diſtance between two other Numbers, then the Ratio of the two former Numbers ſhall be the Duplicate of that Ratio of the two latter. For let the Diſtance IL between the Numbers IK, LM, be double to the Diſtance A *c*, between the Numbers AB, *cd*; and ſince IL is biſected in *l*, we have A *c* = I *l*, = *l*L; and the Ratio of IK to *lm* is equal to the Ratio of AB to *cd*; and ſo the Ratio of IK to LM, the Duplicate of the Ratio of IK to *lm* (*by Def.* 10. *El.* 5.), ſhall be the Duplicate of the Ratio of AB to *cd*.

In like manner, if the Diſtance EL be triple of the Diſtance AC, then will the Ratio of EF to LM be triplicate of the Ratio of AB to CD: For, becauſe the Diſtance is triple, there ſhall be three Times more Proportionals from EF to LM, than there are Terms of the ſame Ratio from AB to CD; and the Ratio of EF to LM, as alſo of AB to CD, is compounded of the equal intermediate Ratios (*by Def.* 5. *El.* 6.) And ſo the Ratio of EF to LM, compounded of three Times a greater Number of Ratios, ſhall be triplicate of the Ratio of AB to CD. So, likewiſe, if the Diſtance GL be quadruple of the Diſtance A *c*, then ſhall the Ratio of GH to LM be quadruplicate of the Ratio of AB to *cd*.

The Logarithm of any Number is the Logarithm of the Ratio of Unity to that Number; or it is the Diſtance between Unity and that Number. And ſo Logarithms expreſs the Power, Place, or Order, which every Number, in a Series of Geometrical Progreſſionals, obtains from Unity. For Example, if there be 10000000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10000000th Place from Unity; then it will be found by

and Computation, that in the same Series from Unity, to 2, there are 3010300 proportional Terms; that is, the Number 2 will stand in the 3010300th Place. In like manner, from Unity to 3, there will be found 4771213 proportional Terms, which Number defines the Place of the Number 3. The Numbers 10000000, 3010300, 4771213, shall be the Logarithms of the Number 10, 2, and 3.

If the first Term of the Series from Unity be called y, the second Term will be y^2, the third y^3, &c. And since the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000} = 10$; also $y^{3010300} = 2$; also $y^{4771213} = 3$; and so on.

Wherefore all Numbers shall be some Powers of that Number which is the first from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Distances of Numbers from Unity, as has been shewn, the Logarithm of Unity shall be 0; for Unity is not distant from itself: But the Logarithms of Fractions are negative, or descending below nothing; for they go on the contrary Way. And so if Numbers, increasing proportionally from Unity, have positive Logarithms, or such as are affected with the Sign +, then Fractions or Numbers, in like manner decreasing, will have negative Logarithms, or such as are affected with the Sign —; which is true when Logarithms are considered as the Distances of Numbers from Unity.

But if Logarithms take their Beginning, not from an integral Unit, but from an Unit that is in some Place of decimal Fractions; for Example, from the Fraction $\frac{1}{10000000000}$; then all Fractions greater than this, will have positive Logarithms; and those that are less, will have negative Logarithms. But more shall be said of this hereafter.

Since in the Numbers continually proportional, CD, EF, GH, IK, &c. the Distances CE, EG, GI, &c. are equal, the Logarithms AC, AE, AG, AI, &c. of those Numbers shall be equidifferent, or the Differences of them shall be equal: And so the Logarithms of proportional Numbers are all in an Arithmetical Progression; and from hence proceeds that common Definition of Logarithms, *viz. that Logarithms*

rithms are Numbers, which, being adjoined to Proportions, have equal Differences.

In the first Kind of Logarithms that *Neper* published, the first Term of the continual Proportionals was placed only so far distant from Unity, as that Term exceeded Unity. For Example, if *vn* be the first Term of the Series from Unity AB, the Logarithm thereof, or the Distance A *n*, or B *y*, was, according to him, equal to *vy*, or the Increment of the Number above Unity. As suppose *v n* be 1,0000001, he placed 0,0000001 for its Logarithm A *n*; and from hence, by Computation, the Number 10 shall be the 23025850th Term of the Series; which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expresses its Distance from Unity in such Parts whereof *v y* or A *n* is one.

But this Position is entirely at Pleasure; for the Distance of the first Term may have any given Ratio to the Excess thereof above Unity; and according to that various Ratio (which may be supposed at Pleasure,) that is, between *vy* and B*y*, the Increment of the first Term above Unity, and the Distance of the same from Unity, there will be produced different Forms of Logarithms.

This first Kind of Logarithms was afterwards changed by *Neper*, into another more convenient one, wherein he put the Number 10 not as the 23025850th Term of the Series, but the 10000000th; and in this Form of Logarithms, the first Increment *v y* shall be to the Distance B *y*, or A *n*, as Unity, or AB, is to the Decimal Fraction 0,4342994, which therefore expresses the Length of the Subtangent AT, *Fig*. 4.

After *Neper*'s Death, the excellent Mr. *Henry Briggs*, by great Pains, made and published Tables of Logarithms according to this Form. Now since in these Tables, the Logarithm of 10, or the Distance thereof from Unity, is 1,0000000, and 1, 10, 100, 1000, 10000, &c. are continual Proportionals, they shall be equidistant. Wherefore the Logarithm of the Number 100 shall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10000 shall be 4,0000000; and so on.

Hence the Logarithms of all Numbers between 1 and 10 must begin with 0, or 0 must stand in the

first

and [illegible]ce to the Left-hand; for they are lesser than [illegible] Logarithm of the Number 10, whose Beginning is the Log[illegible] and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and less than 2,0000000. Also the Logarithms between 100 and 1000 begin with 2; for they are greater than the Logarithm of 100 which begins with 2, and less than the Logarithm of 1000 that begins with 3. In the same manner it is demonstrated, that the first Figure to the Left-hand of the Logarithms between 1000 and 10000 must be 3; and the first Figure to the Left-hand of the Logarithms between 10000 and 100000 will be 4; and so on.

The first Figure of every Logarithm to the Left-hand is called the Characteristic, or Index, because it shews the highest or most remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the highest or most remote Place from Unity of the correspondent Number, to the Left-hand, will be the Place of Tens. If the Index be 2, the most remote Figure of the correspondent Number shall be in the second Place from Unity, that is, it shall be in the Place of Hundreds; and if the Index of a Logarithm be 3, the last Figure of the Number answering to it, shall be in the Place of Thousands. The Logarithms of all Numbers that are in decuple or subdecuple Progression, only differ in their Characteristics, or Indices, they being written in all other Places with the same Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17000, are the same, unless in their Indices; for since 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17000; therefore the Distances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17000, shall be all equal. And so, since the Distance between 1 and 17, or the Logarithm of the Number 17 is 1.2304489, the Logarithm of the Number 170 will be=2.2304489, and the Logarithm of the Number 1700 shall be 3.2304489, because the Logarithm of the Number 100=2.0000000. In like manner, since the Logarithm of the Number 1000=3.0000000, the Logarithm of the Number 17000 shall be 4.2304489.

So also the Numbers, 6741. 674,8. 67,48. 6,748. 0,6748. 0,06748, are continual Proportionals in the Ratio of 10 to 1; and so their Distances from each other shall be equal to the Distance or Logarithm of the Number 10, or equal to 1,0000000. And so, since the Logarithm of the Number 6748 is 3,8291751, the Logarithms of the other Numbers shall be as in the Margin; where you may observe, that the Indices of the last two Logarithms are only negative, and the other Figures positive; and so, when those other Figures are to be added, the Indices must be subtracted, and contrariwise.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	—1,8291751
0,06748	—2,8291751

CHAP. II.

Of the Arithmetic of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BEcause, in Multiplication, Unity is to the Multiplier; as the Multiplicand is to the Product; the Distance between Unity and the Multiplier shall be equal to the Distance between the Multiplicand and the Product. If therefore the Number GH be to be multiplied by the Number EF, the Distance between GH and the Product must be equal to the Distance AE, or to the Logarithm of the Multiplier; and so, if GL be taken equal to AE, the Number LM shall be the Product; that is, if the Logarithm of the Multiplicand AG be added to the Logarithm of the Multiplier AE, the Sum shall be the Logarithm of the Product.

In Division, the Divisor is to Unity, as the Dividend is to the Quotient; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and the Quotient. So if LM be to be divided by EF, the Distance EA shall be equal to the Distance between LM and the Quotient; and

and so, if LG be taken equal to EA, the Quotient will be at G; that is, if from AL, the Logarithm of the Dividend, be taken GL, or AE, the Logarithm of the Divisor, there will remain AG, the Logarithm of the Quotient.

And from hence it appears, that whatsoever Operations in common Arithmetic are performed by multiplying or dividing of great Numbers, may be done much easier, and more expeditiously, by the Addition or Subtraction of Logarithms.

For Example, Let the Number 7589 be to be multiplied by 6757. Now, if the Logarithms of those Numbers be added together, as in the Margin, their Sum will be the Logarithm of the Product, whose Index 7 shews, that there are seven Places of Figures, besides Unity, in the Product; and in seeking this Logarithm in Tables, or the nearest equal to it, I find that the Number answering thereto, which is lesser than the Product, is 51278000; and the Number greater than the Product is 51279000; and if the adjoined Differences, and proportional Parts, be taken, the Numbers that must be added to the Place of Hundreds and Tens in the Product are 87; and that which must be added in the Place of Unity, will necessarily be 3, since seven Times 9=63; and so the true Product shall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers to be added in the Place of Tens or Hundreds could not be had from those Tables of Logarithms which consist of but 7 Places of Figures, besides the Characteristic; and so, in this Case, the *Valacquian* or *Briggian* Tables should be used; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

Log.	3.8801846
Log.	3.8297539
Log.	7.7099385

If the Number 78596 be to be divided by 276, by subtracting the Logarithm of the Divisor from the Logarithm of the Dividend, the Logarithm of the Quotient willl be had. And to this Logarithm, the Number 282, 719 answers; which therefore shall be the Quotient.

Log.	4.8954004
Log.	2.4440448
Log.	2.4513556

Because Unity, any assumed Number, the Square thereof, the Cube, the Biquadrate, &c. are all con-

tinual Proportionals, their Distances from each other shall be equal to one another. And so it is manifest, that the Distance of the Square from Unity is double of the Distance of its Root from the same: Also the Distance of the Cube is triple of the Distance of its Root; and the Distance of the Biquadrate is quadruple of the Distance of its Root from Unity, &c. And so, if the Logarithm of any Number be doubled, we shall have the Logarithm of its Square; if it be tripled, we shall have the Logarithm of its Cube; and if it be quadrupled, the Logarithm of its Biquadrate. And contrariwise, if the Logarithm of any Number be bisected, we shall have the Logarithm of the Square Root thereof: Moreover, a third Part of the said Logarithm will be the Logarithm of the Cube Root of the Number; and a fourth Part, the Logarithm of the Biquadrate Root of that Number.

Hence, the Extractions of all Roots are easily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square Root of 5, the Half of 0,6989700 must be taken, and then that Half 0,3494850 will be the Logarithm of the Square Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2,236068 nearly answers.

CHAP. III.

Of the Arithmetic of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is necessary, for avoiding the Trouble of adding one Part of a Logarithm, and subtracting the other, that Logarithms do not begin from an integral Unit, but from some Unit that is in the Tenth or Hundredth Place of Decimal Fractions: For Example, let PO be $\frac{1}{10000000000}$, and from this let the Logarithm begin. Now this Fraction is ten Times more distant from Unity to the Left-hand, than the Number 10 is distant therefrom to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to PO. And so, if AB be Unity, the

the Logarithm thereof, according to this Suppofition, will not be 0, but QA=10.0000000. Now the Diftance of Ten from Unity is 1.0000000, whence the Diftance of the Number 10 from PO will be 11.0000000. Alfo the Diftance of the Number 100 from PO, or its Logarithm, beginning from PO, fhall be 12.0000000; and the Logarithm of 1000, or the Diftance from PO, will be 13.0000000. And thus, the Indices of all Logarithms are augmented by the Number 10; and thofe Fractions whofe Indices are —1, or—2, or—3, &c. are now made 9, 8, or 7, &c.

But if Logarithms begin from the Place of a Fraction, whofe Numerator is Unity, and Denominator Unity with 100 Cyphers added to it (which they muft do when Fractions occur that are lefs than PO), then that Fraction will be 100 Times more diftant from Unity, than 10 is diftant from it; and fo the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and fo on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than PO (whereat they begin) will be pofitive. And fince the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. are in a continual Geometrical Progreffion, they will be equally diftant from each other; and accordingly their Logarithms will be equidifferent: And fo, when the Logarithm of 10 is 11.0000000, and the Logarithm of Unity is 10.0000000; then the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000; and, in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Alfo, for the fame Reafon, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of the Logarithm of the Fraction $\frac{1}{1000}$ fhall be 97, &c. And thefe Indices fhew in what Place from Unity the firft Figure of the Fraction, not being a Cypher, muft be put. For Example, if the Index be 4, the Diftance thereof from the Index of Unity (which is 10), *viz.* 6, fhews that the firft fignificative Figure of the Decimal is in the fixth Place from Unity; and therefore five Cyphers are to be perfixed

thereto towards the Left-hand. So, also, if the Index of Unity be 100, and the Index of the Fraction be 80, the first Figure thereof shall be in the 20th Place from Unity, and 19 Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction GH by the Fraction DC. Because Unity is to the Multiplier as the Multiplicand is to the Product; the Distance between Unity and the Multiplier shall be equal to the Distance between the Multiplicand and the Product. Therefore, if there be taken GI=AC, the Product IK shall be at I. And, accordingly, if from OG, the Logarithm of the Multiplicand, there be taken GI or AC, there will remain OI, the Logarithm of the Product. But AC=OA—OC, which taken from OG, there will remain OG+OC—OA=OI; that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity (which is always expressed by 10 or 100 with Cyphers), the Logarithm of the Product will be had. For Example, let the Decimal Fraction 0,00734 be to be multipiled by the Fraction 0,000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margin; which being added together, and the Logarithm of Unity being taken away from the Sum, the Remainder is the Logarithm of the Product, whose Index 94 shews, that the first Figure of the Product is in the sixth Place from Unity; and so there must be five Cyphers prefixed, and then the Product will be 0,0000642984.

97,8656961
96,9425041
94,8082002

In Division, the Divisor is to Unity, as the Dividend is to the Quotient; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and Quotient. And so, if the Fraction IK be to be divided by DC, you must take G=CA, and the Place of the Quotient shall be G. But CA=OA—OC, which being added to OI, we have OA+OI—OC=OG; that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Divisor, there will remain the Logarithm of the Quotient; so if the Number CD be to be

be divided by IK, you must take the Distance CS= IA, and then ST will be the Quotient, whose Logarithm is OA+OC—OI. Let CD=0,347, IK, =0.00478. Then add the Logarithm of Unity to the Logarithm of CD; that is, put 1 or 10 before the Index thereof, and from that subtract the Logarithm of the Divisor, and the Remainder will be the Logarithm of the Quotient, whose Index 1 shews, that the Quotient is between the Numbers 10 and 100; and I seek the Number answering the Logarithm, which I find to be 72,594.

19.5403295
7.6794279
11.8609016

If the Logarithm of a Vulgar Fraction, for Example $\frac{7}{8}$, be required, the Logarithm of Unity must be added to the Logarithm of the Numerator 7; or, which is all one, you must put 10 or 100 before the Index thereof, and subduct from it the Logarithm of the Denominator 8; and there will remain the Logarithm of the Vulgar Fraction $\frac{7}{8}$, or the Decimal .875.

10.8450980
0.9030900
9.9420080

If the Powers of any Fraction DC be required, you must assume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually proportional from Unity. Besides, AE= 2 AC=2 AO—2 OC; whence OE=OA—AE =2 OC—OA; that is, the Logarithm of the Square is the Double of the Logarithm of the Root, less the Logarithm of Unity. In like manner, since AG= 3 AC= 3 OA—3 OC, we shall have OG=OA— AG= 3 OC—2 OA= the Logarithm of the Cube, = triple the Logarithm of the Root,—the Double of the Logarithm of Unity. For the same Reason, because AI = 4 AC = 4 OA—4 OC, we have OI = 4 OC—3 OA, which is the Logarithm of the Biquadrate. And, universally, if the Power of a Fraction be n, and the Logarithm L, then shall the Logarithm of the Power n=n L—n OA+OA; that is, if the Logarithm of a Fraction be multiplied by n, and from the Product be taken the Logarithm of Unity, multiplied by n—1, the Logarithm of the Power n of that Fraction will be had.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{20}$=,05, the Logarithm of this Fraction 8.6989700, which, being multiplied by 6, gives the Number 52.1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, *viz.* 2.1938200, to which the Number ,000000015625 answers. For the Index 2 shews, that 7 Cyphers must be put before the first Figure,

If the 8th Power of the Fraction ,05 be required, by multiplying the Logarithm by 8, there will be produced 69.5917600; and since 70, which is seven Times the Index of the Logarithm of Unity, cannot be taken from 69, unless we run into negative Numbers, the Index of the Logarithm of Unity must be supposed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm drawn into 8, gives 789,5917600; and if 700, which is 7 Times the Index of the Logarithm of Unity, be taken from 789, there will remain 89.5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whose correspondent Number is ,000000000039062. For since the Index is 89, and the Difference thereof from 100 to 11; the first significative Figure of the Fraction shall be in the 11th Place from Unity; and so there must be 10 Cyphers placed before it.

If the Roots of the Powers of Fractions be desired, for Example, the Square Root of the Fraction EF; because the Root is a mean Proportional between the Fraction and Unity, you must bisect AE in C, and then CD will be the Square Root of the Fraction EF. But $AC = \frac{1}{2} AE = \frac{OA-OE}{2}$; and so the Logarithm of the Root $= OA-AC = \frac{OA+OE}{2}$. And if the Cube Root of the Fraction GH be sought, this shall be the first of two mean Proportionals between Unity and GH; and so, if AG be divided into three equal Parts, the first of which is AC, then CD shall be the Root sought: And because $AC = \frac{1}{3} AG = \frac{OA-OG}{}$, if this be taken from OA, there will remain

main $\frac{2OA+OG}{3} = OC$ = Logarithm of the Cube Root of the Fraction GH. So, likewise, the Biquadrate Root of the Fraction IK will be had, by dividing AI into four equal Parts; for the Root is the first of three mean Proportionals between Unity and the Fraction; and, consequently, if $AC = \frac{1}{4}AI$, then will CD be the Biquadrate Root of the Fraction IK. But $AC = \frac{1}{4}AI = \frac{OA-OI}{4}$; and so $OC = OA-AC = \frac{3OA+OI}{4}$.

And universally, if the Root of any Power n of the Fraction LM be required, the Logarithm of the Root thereof will be $\frac{nOA-OA+OL}{n}$; that is, if the Number $n-1$ be prefixed to the Index of the Logarithm, and the Logarithm thus augmented be divided by n, the Quotient will give the Logarithm of the Root sought. So if the Cube Root of the Fraction $\frac{1}{2}$ or ,5 be sought, you must place $2=n-1$ (since the Cube Root is required) before the Logarithm thereof, and there will be had 29.6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$; and the Number ,7937, answering to this Logarithm, is the Root sought.

CHAP. IV.

Of the Rule of Proportion by Logarithms.

THE Rule of Proportion shews how, by having three Numbers given, a fourth Proportional to them may be found; *viz.* if the second and third Terms be multiplied by one another; and the Product divided by the first Term, then will the Quotient be the fourth proportonal Term sought. But this fourth Term is much easier found by Logarithms; for if the Logarithm of the first Term be taken from the Sum of the Logarithms of the second and third Term, the Number

ber remaining will be the Logarithm of the fourth, sought.

Or this may be done something easier yet, if instead of the Logarithm of the first Term be taken its Complement Arithmetical, or the Difference of that Logarithm, and the Number 10.0000000, which is done by setting down the Difference between each Figure of the Logarithm, and the Figure 9; for then, if that Arithmetical Complement be added to the Sum of the other two Logarithms; and if Unity, which is the first Figure to the Left-hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term sought; and so, by this Way, the Logarithm of the fourth Term is found by only one Addition of three Numbers. The Reason of this will be manifest from hence: Let there be three Numbers A, B, C, the first of which is to be taken from the Sum of the second and third. Now this may not only be done by the common Way, but, likewise, if there be any other Number E taken, and from this there be taken A, there will remain E—A; and if the Numbers B, C, and E—A, be all added together, and from their Sum be taken E, there will remain B+C—A. So, if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100 which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

	85
	23
	108

Here follow some Trigonometrical Examples of the Rule of Proportion solved by Logarithms.

Let ABC be a Right-lined Triangle, wherein are given the Angle A 36 Degrees 46′, the Angle B 98 Degrees 32′, and the Side BC 3478; the Side AC is required. Say (*by Case* 1. *of Plane Trig.*), as the Sine of the Angle A is to the Sine of the Angle B, so is BC to AC. And because the Logarithm Sine of the Angle A is the first Term of the Analogy, I substitute its Complement Arithmetical for the same, and add the Logarithm of BC, the Logarithm of S, B, and the said Complement, all three together, and reject

Arith. Comp. S, A.	0.2228938
Log. Sin. B.	9.9951656
Log. BC.	3.5413296
Log. AC.	13.7593890

 Unity,

Unity, which is in the first Place to the Left-hand; and then the Logarithm of the Side AC will be given, and the Number answering thereto is 5746,306, equal to the Side sought AC.

Let there be a Spherical Triangle ABC, in which are given all the Sides, *viz.* BC = 30 Degrees, AB = 24 Degrees 3′, and AC = 42 Degrees 8′; the Angle B is required. Let BA be produced to M, so that BM = BC; then will AM, the Difference of the Sides BC, BA, be equal to 5 Degrees 57′. Now (*by Case* 11. in oblique-angled Spherical Triangles) say, As the Rectangle under the Sines of the Legs is to the Square of Radius, so is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$, to the Square of the Sine of one half the Angle B.

But $\frac{AC+AM}{2} = 24$ Degrees 2′, and $\frac{AC-AM}{2}$ = 18 Degrees 6′; and because the first Term of the Analogy is the Rectangle under the Sines of AB, BC, and the second Term is the Square of Radius, the Sum of the Logarithm Sines of AB, BC, must be taken from double the Logarithm of Radius, and what remains must be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$, which is the same as if the Logarithm Sines of each of the Arcs AB, BC, were subtracted from the Log. of Radius; or if the Complements Arithmetical of these Sines be taken, and

Log. S, BC Comp. Arith.	0.3010299
Log. S, AB Comp. Arith.	0.3898364
Log. S, $\frac{AC+AM}{2}$	9.6098803
Log. S, $\frac{AC-AM}{2}$	9.4923083
2 Log. S $\frac{1}{2}$ Angle B	19.7930549

those Complements and the said Sines be all added together, then shall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And so the Half thereof, *viz.* the Logarithm 9.8965274, is the Logarithm Sine of half the Angle B = 51 Degrees 59′ 56″, and the Double of this Angle shall be 103 Degrees 59′ 52″ = B, which was sought.

CHAP.

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms, any Term in a Series of Proportionals, either increasing or decreasing. Fig. 3.

IF any where in the Axis of the Logarithmical Curve, there be taken any Number of equal Parts SV, VY, YQ, &c. and at the Points S, V, Y, Q, &c. be raised the Perpendiculars ST, VX, YZ, QΠ, &c. then, from the Nature of the Curve, shall all these Perpendiculars be continually proportional; and therefore, also, the continual Increments Xx, Zz, $\Pi\pi$, shall be proportional to their Wholes. For since ST : VX : : VX : YZ : : YZ : QΠ, it shall be (by Division of Proportion) ST : Xx : : VX : : Zz : : YZ : $\Pi\pi$; and (by Composition of Proportion) VX : Xx : : YZ : Zz : : QΠ : $\Pi\pi$. Hence if Xx be any Part of any Right Line ST, then will Zz be the same Part of the Right Line VX, and also $\Pi\pi$ the same Part of the Right Line YZ. For Example, if Xx be the $\frac{1}{20}$ Part of ST, then will $Zz = \frac{1}{20}$ VX, and $\Pi\pi = \frac{1}{20}$ YZ; or, which comes to the same, we shall have $VX = ST + \frac{1}{20} ST$, $YZ = VX + \frac{1}{20} VX$, also, $Q\Pi = YZ + \frac{1}{20} YZ$.

Now make, as ST is to VX, so is Unity AB to NR; then shall AN = SV; and so each of the Right Lines SV, VY, YQ, &c. shall be equal to the Logarithm of RN; and AV, the Logarithm of the Term VX, shall be equal to AS + AN = Logarithm of ST + Logarithm of NR. Also AY, the Logarithm of the Term YZ, shall be equal to AS + 2 AN = Logarithm ST + 2 Logarithm NR; and AQ, the Logarithm of the Term QΠ, shall be equal to AS + 3 AN = Logarithm ST + 3 Logarithm NR. And universally, if the Logarithm of the Number NR be multiplied by a Number, expressing the Distance of any Term from the first, and the Product be added

is the Logarithm of the first Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreasing, that is, if the Terms diminish in a continual Ratio, and Q π be the first Term; then the Logarithm of any other will be had, by multiplying the Logarithm of the Number NR, by a Number that expresses the Distance of its Term from the first, and subtracting the Product from the Logarithm of the first. And if the said Product be greater than the Logarithm of the first Term, then the Logarithms must begin from a Unit in some Place of Decimal Fractions, as from QP, and then the Logarithm of the Number Q π will be OQ.

Now, let LM represent any Money, or Sum of Money, put out to Interest, so that the Interest thereof be accounted but at the End of every Year, and let K *k* be the Gain or Interest thereof at the End of the first Year; then will IK be the Sum of the Interest and Principal. And again IK, becoming the Principal at the End of the first Year, H *h*, which is proportional to IK, or in a constant Ratio, will be the Gain at the End of the second Year; and so HG, at the End of the second Year, will become the Principal; and at the End of the third Year F *f*, proportional to GH, will be the Gain. Now let us suppose the Principal to be augmented every Year $\frac{1}{20}$ Part thereof, so that IK $=$ LM $+ \frac{1}{20}$ LM, GH $=$ IK $+ \frac{1}{20}$ IK, EF $=$ GH $+ \frac{1}{20}$ GH, and so on. And accordingly, the Terms LM, IK, GH, EF, *&c.* are continual Proportionals, and it is required to find the Amount of the Money at the End of any Number of Years.

Let LM be a Farthing. Because LM is to IK as 1 to $1 + \frac{1}{20}$, or as 1 to 1.05, as AB is to NR, then will NR $=$ 1.05, whose Logarithm AN, is 0.0211893, or, more accurately, 0.021892991, it is required to find the Amount of a Farthing, put out at Compound Interest, at the End of 600 Years. Multiply AN by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$, *viz.* 97.0177288 (for a Farthing is $\frac{1}{960}$ Part of a Pound), and the Sum 109.7313082 shall be the Logarithm of the Number sought; and since the Index 109 exceeds the Index of Unity by 9, there shall be nine Places of Figures above Unity in the correspondent Number;

ber; and that Number, being sought in the Table, will be found greater than 5386500000, and less than 5386600000. And therefore a Farthing put out, at Interest upon Interest, at 5 per Cent. per Annum, at the End of 600 Years, will amount to above 5386500000 Pounds; which Sum could hardly be made up, by all the Gold and Silver that has been dug out of the Bowels of the Earth, from the Beginning of the World to this Time.

Let QΠ expound any Sum of Money due to some Person at the End of a full Year. Now it is certain, that if the Debtor should pay down, at present, the whole Sum of Money, he would lose the yearly Usury or Interest that his Money would gain him; and so a lesser Sum, being put out to Interest, will, at the End of one Year, together with the Interest thereof, be equal to the Sum of Money QΠ. Now this present Sum of Money, which, together with the Interest thereof, is equal to the Sum of Money QΠ, is called the present Worth of the Money QΠ. Let AN be the Logarithm of the Ratio which the Principal has to the Sum of the Principal and Interest, that is, if the Principal be twenty Times the yearly Interest, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QY equal to AN; then will AY be the Logarithm of the present Worth of the Money QΠ. For it is manifest, that the Money YZ put out to Interest, will, at the End of one Year, amount to the Money QΠ; and so, to have the Logarithm of the present Worth thereof, or YZ, the Logarithm AN must be taken from the Logarithm AQ, and there will remain the Logarithm AY of the present Worth, or YZ. But if the Sum QΠ be not due till the End of two Years, then the Logarithm 2AN must be subtracted from the Logarithm AQ, and there will remain AV, the Logarithm of the present Worth, or of the Sum that must be paid at present for the Money QΠ due at the End of two Years. For it is manifest, that the Money VX being put out to Interest, will, at the End of two Years amount to the Sum of Money QΠ. By the same Reason, if the Sum QΠ be not due until the End of three Year, the Logarithm 3AN must be subtracted from the Logarithm of QΠ, and the Remainder AS shall be the Logarithm of the Number ST, or

ST

I' shall be the present Worth of the Sum $Q\pi$ due the three Years End. And universally, if the Logarithm AN be multiplied by the Number of Years, the End of which the Sum $Q\pi$ is due, and the ... produced be taken from the Logarithm AQ, ... the Logarithm of the present Worth of the ... And from hence it is manifest, if 5386500000 Pounds be due to some Society at the End of 600 Years, then would the present Worth of that vast Sum of Money be scarcely a Farthing.

If the proportional Right Lines HG, EF, AB, CD, *Fig*. 4. are Ordinates to the Axis of the Logarithmical Curve, and if their Ends FH, DB, be joined by Right Lines, which, produced, meet the Axis in the Points P and K, then the Right Lines GP, AK, will be always equal. For since GH : EF : : AB : CD; it will be, as GH : F*s* : : AB : DR, But because of the equiangular Triangles PGH, H*s* F, as also KAB, BRD, we have PG : H*s* : : (GH : F*s* : : AB : DR : :) KA : BR. And since the Consequents H*s*, BR, are equal, the Antecedents PG, KA, shall be also equal. W. W. D.

If the Right Lines CD, EF, equally accede to AB, GH, so that the Point D at last may coincide with B, and the Point F with H, then the Right Lines DBK, FHP, which did cut the Curve before, will be changed into the Tangents BT, HV. And the Right Lines AT, GV, will be always equal to each other; that is, the Portion of the Axis AT, or GV, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every-where be of a constant and given Length. And this is one of the chief Properties of the Logarithmical Curve; for the different Species or Forms of those Curves are determined by the Subtangents.

The Logarithms, or the Distances from Unity of the same Number, in two Logarithmical Curves of different Species, will be proportional to the Subtangents of their Curves. For let HBD, SNY, *Fig*. 4, 5. be Curves, whose Subtangents are AT, MX, and let AB = MN = Unity; also, DC = QY; then shall AC, the Logarithm of the Number CD, in the Logarithmical Curve HD, be to MQ, the Logarithm of the Number QY (or of the said CD), in the Curve SY, as the Subtangent AT is to the Subtangent MX.

MX. For let there be supposed an infinite Number of mean Proportional Terms between AB, CD, to MN, QY, in the Ratio of AB to ab, or MN to mn; and since AB = MN, then will $ab = mn$, and also $bc = no$. And because the Number of proportional Terms in each Figure are equal, they divide by and the Lines AC, MQ, into equal Numbers of the Bows, the first of which are Aa, Mm, and so the said Parts shall be proportional to their Wholes; that is, it will be as Aa : Mm : : AC : MQ. And because the Triangles TAB, Bcb, are similar (for the Part of the Curve Bb nearly coincides with the Portion of the Tangent), as also the Triangles XMN, Non, we have Aa, or Bc : bc : : TA : AB.

Also, as no, or bc : No : : MM, or AB : MX.

Where (by Equality of Proportion) it will be, Bc : No : : TA : MX : : Aa : Mm : : AC : MQ; *which was to be demonstrated.* If AT be called a, since AB : AT : : bc : Bc, then will B$c = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a small Matter exceeding it, be given, then will the Subtangent of the Logarithmical Curve be had. For the Excess bc is to the Logarithm Bc, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference shall be to the Difference of the Logarithms, as one of the Numbers is to the Subtangent. For Example, if the Increment bc be ,00000 00000-00001 02255 31945 60259, and Bc or Aa the Logarithm of the Number ab be ,00000 00000 00000-44408 92098 50062. Now if a fourth Proportional be found to the said two Numbers and Unity, *viz.* 43429448190325 1, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve expressing *Briggs*'s Logarithms.

If a Sum of Money be put out to Interest on this Condition, that a proportional Part of the yearly Rate of Interest thereof be accounted every Moment of Time, *viz.* so that at the End of the first Moment of Time, or indefinitely small Particle of a Year, the Interest gotten thereby be proportional to that Time; which being added to the Principal, again begets Interest at the End of the second Moment of Time,

and

and then the Principal and this Interest become a Principal, and so on; it is required to find the Amount of that Sum at the Year's End. Let a be nearly the ... of Unity, or of one Pound. Then, if one Year, or 1, gives the Interest a, the indefinitely ... of a Year Mm will give the Interest ... proportional to Mm; and, accordingly, if Unity be expounded by MN, the first Increment thereof shall be $no = Mm \times a$. This being granted, let a Logarithmical Curve be supposed to be described through the Points $N n$, whose Axis is OMQ. Then, in this Curve, if the Proportion of the Axis MQ expresses the Time, the Ordinate QY will represent the Money proportionally increasing every Moment, to that Time. For if there be taken ml, &c. $= Mm$, the Ordinates lp, &c. shall be a Series of continual Proportionals in the Ratio of MN to mn; that is, they increase in the same Ratio as the Money doth.

Again, Let the Right Line NX touch the Logarithmical Curve in N, and the Subtangent thereof MX shall be constant and invariable, and the small Triangle Non shall be similar to the Triangle XMN. But it has been proved, that the Increment $no = Mm \times a = No. \times a$; and so $no : No :: No \times a : No :: a : 1$. But as no is to Na, so shall MN be to MX. Wherefore it shall be, as a is to 1, so is MN, or 1, to $MX = \frac{1}{a} =$ Subtangent.

Now if the yearly Rate of Interest be $\frac{1}{20}$ Part of the Principal, or if $a = \frac{1}{20} = .05$, then will $MX \times \frac{1}{a} = 20$.

Because in different Forms of Logarithms, the Logarithms of the same Number are proportional to the Subtangents of their Curves: If MQ expresses the Time of a whole Year, or Unity, then shall QY be the Amount of the Money at the Year's End. And to find QY, say, As MX, or 20, is to 0.4342944 (which Number expounds the Subtangent of the Logarithmical Curve expressing *Briggs*'s Logarithms), so is one Year, or Unity, to a *Briggian* Logarithm answering to the Number QY. This Logarithm will be found 0.0217147, and the Number answering to the same is $1.05127 = QY$, whose Increment above

Unity, or the Principal, exceeds the yearly Intereſt ,05 but a ſmall Matter. And ſo if the yearly Intereſt of 100 Pounds, be 5 Pounds, the proportional yearly Intereſt, which is added to the Principal at the End of each Particle of the Year, will only at the Year's End to 5 Pounds 2 Shill[illegible] Pence.

And if ſuch a Rate of Intereſt be required, that every Moment a Part of it continually proportional to the increaſing Principal be added to the Principal, ſo that at the Year's End an Increment be produced that ſhall be any given Part of the Principal; for Example, the $\frac{1}{20}$ Part; ſay, As the Logarithm of the Number 1.05 is to that [illegible], as 0.0211893 is to 1; ſo is the Subtangent 0,4342944 to $\frac{1}{a} = 20.49$, and then will $a = \frac{1}{20.49} = .0488$. For if ſuch a Part of the Rate of Intereſt .0488 be ſuppoſed, as anſwers to a Moment, that is, having the ſame Ratio to .0488 as a Moment has to a Year, and it be made, as Unity to that Part of the Rate of Intereſt, ſo is the Principal to the momentaneous Increment thereof; then will the Money, continually increaſing in that Manner, be augmented, at the Year's End, the $\frac{1}{20}$ Part thereof.

CHAP. VI.

Of the Method by which Mr. Briggs *computed his Logarithms, and the Demonſtration thereof.*

ALthough Mr. *Briggs* has no-where deſcribed the Logarithmical Curve, yet it is very certain, that, from the Uſe and Contemplation thereof, the Manner and Reaſon of his Calculations will appear. In any Logarithmical Curve HBD, let there be three Ordinates AB, *ab*, *qs*, nearly equal to one another; that is, let their Differences have a very ſmall Ratio to the ſaid Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For ſince the Ordinates are nearly equal to one another, they will be very nigh to

to each other; and so the Part of the Curve B s, intercepted by them, will almost coincide with a strait Line: for it is certain, that the Ordinates may be so ... to each other, that the Difference between the ... the Curve, and the Right Line subtending it, ... to that Subtense a Ratio less than any given ... Therefore the Triangles B c b, B r s, may be taken for Right-lined; and will be equiangular. Wherefore, as $sr : bc : Br : Bc :: Aq : Aa$; that is, the Excesses of the Ordinates, or Lines above the least, shall be proportional to the Differences of their Logarithms. And from hence appears the Reason of the Correction of Numbers and Logarithms by Differences and proportional Parts. But if AB be Unity, the Logarithms of Numbers shall be proportional to the Differences of the Numbers.

If a mean Porportional be found between 1 and 10, or, which is the same Thing, if the Square Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof shall be $\frac{1}{2}$ of the Logarithm of 10, and so will be given. If, again, between the Number before found, and Unity, there be found a mean Proportional, which may be done by extracting the Square Root of the said Number, this Number or Root, will be twice nearer to Unity than the former, and its Logarithm will be one Half of the Logarithm of that, or one Fourth of the Logarithm of 10. And if in this manner the Square Root be continually extracted, and the Logarithms bisected, you will at last get a Number, whose Distance from Unity shall be less than the $\frac{1}{1000000000000000000}$ Part of the Logarithm of 10. And after Mr. *Briggs* had made 54 Extractions of the Square Root, he found the Number 1,00000 00000 00000-12781 91493 20032 3442; and its Logarithm was 0. 00000 00000 00000 05551 11512 31257 82702. Suppose this Logarithm to be equal to A q, or B r, and let q s be the Number found by extracting the Square Root; then will the Excess of this Number above Unity, *viz.* r s = ,00000 00000 00000 12781 91493-20032 3442.

Now, by means of these Numbers, the Logarithms of all other Numbers may be found in the following manner: Between the given Number (whose Loga-

rithm is to be found) and Unity, find so many mean Proportionals (as above), till at last a Number be gotten so little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of significative Figures after those. Let this Number be ab, and let the significative Figures, with the Cyphers prefixed [illegible] denote the Difference bc. Then say, As [illegible] Bow [illegible] rence rs is to the Difference bc, so is Br a given Logarithm, to Bc, or Aa, the Logaritom of the Number ab; which therefore is given. And if this Logarithm be continually doubled, the same Number of Times as there were Extractions of the Square Root, you will at last have the Logarithm of the Number sought. Also, by this Way may the Subtangent of the Logarithmical Curve be found, *viz.* by saying, As $rs : Br :: AB$, or Unity : AT, the Subtangent, which therefore will be found to be 0.434294482903251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards, as also its Logarithm, and the Logarithm of another Number, sufficiently near to NM, be sought, say, As NM is to the Subtangent XM, so is no, the Distance of the Numbers, to No, the Distance of the Logarithms. Now, if NM be Unity $=$ AB, the Logarithms will be had by multiplying the small Differences bc by the constant Subtangent AT.

By this Way may be found the Logarithms of 2, 3, and 7; and by these the Logarithms of 4, 8, 16, 32, 64, *&c.* 9, 27, 81, 243, *&c.* as also 7, 49, 343, *&c.* And if from the Logarithm of 10 be taken the Logarithm of 2, there will remain the Logarithm of 5; so there will be given the Logarithms of 25, 125, 625, *&c.*

The Logarithms of Numbers compounded of the aforesaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, *&c.* are easily had by adding together the Logarithms of the component Numbers.

But since it was very tedious and laborious to find the Logarithms of the prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second, and third, *&c.* Differences; therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and, lastly, Dr. *Halley*, have published infinite converging Series, by which the Logarithms of Numbers

Numbers to any Number of Places may be had more expeditiously, and truer: Concerning which Series Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*; wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, easily and expeditiously, the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Numbers $z-1$ and $z+1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms, will be had, which let be called y. Therefore, also, the Logarithm of a Number, which is a Geometrical Mean between $z-1$ and $z+1$, will be given, *viz.* equal to the Half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{151120z^7} + \frac{13}{25220z^9}$, &c. shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z-1$ and $z+1$ has to the Arithmetical Mean, *viz.* to the Number z.

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But, if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example, it is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2 with the Index 4 prefixed to it; and the Difference of the Logarithms of 20000 and 20002 is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.00004 34272 7687. And if this Difference be divided by $4z$, or 80004, the Quotient $\frac{y}{4z}$

 shall

shall be - - - - - - - - - -	0.00000 00005 42814
And if the Logarithm of the Geometrical Mean be added	4.30105 17093 02416
to the Quotient, the Sum will	4.30105 17098 45230

be the Logarithm of 20001. Wherefore it is manifest, that to have the Logarithm to 14 Places of Figures, there is no Necessity of continuing out the Quotient beyond six Places of Figures. But if you have a mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlaq*'s Table, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will mostly consist in setting down the Numbers.

Note, This Series is easily deduced from that found out by Dr. *Halley*; and those who have a mind to be informed more in this Matter, let them consult his above-named Treatise.

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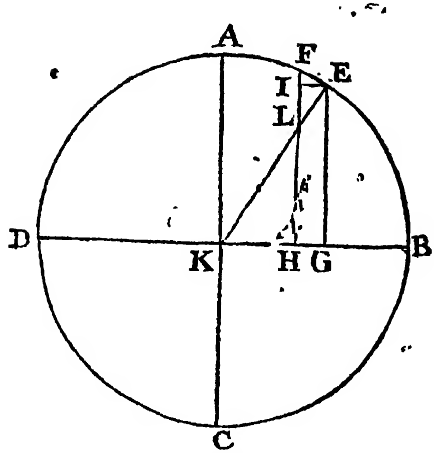
APPENDIX.

IT is needlefs here to write a Prefatory Difcourfe, fetting forth the Ufe and Invention of Logarithms, fince the Author has fupplied that, in his Preface to the Treatife of the Nature and Arithmetic of Logarithms annexed to thefe Elements: It is enough to inform the Reader, that my chief Defign in writing this *Appendix* was, to render their Conftruction eafy, by invefting various Theorems for that Purpofe, and illuftrating them by proper Examples; all which is performed in the actual Operation of making the Logarithms of the firft 10 Numbers, and of the prime Number 101, which is more than fufficient to inform the meaneft Capacity how to examine or conftruct the whole Table. I have alfo fhewn how, from the Logarithm given, to find its correfponding Number; and the Inveftigation of the Series omitted by the Author in Page 357, for expeditioufly finding the Logarithms of large Numbers. As to thofe Series exhibited by him in his Trigonometrical Treatife, Page 287, for making the Sines and Cofines; I muft declare, that I have exceeded my firft Intentions, which were to give their Inveftigation only; but confidering, that as they depended upon the *Newtonian* Series without the Inveftigation of which our Author's Series could never be thoroughly underftood; I thought it would therefore prove acceptable, if I fhewed their Inveftigations too, from which thofe of our Author eafily flow. In order to which, and to keep the Reader no longer in Sufpenfe, let r be put for the Radius KE of the Circle ABCD; a for the Arc BE, whofe Length is to be inveftigated; and s equal to the Sine EG: Then is FE $=a$, and IF $=s$.



Then, the Triangles KGE, KHL, are similar, because LH is parallel to EG; the Triangles KLH, FLE, are similar, because the Angles KLH, FLE, are equal, and the Angles KHL, FEL, are both Right Angles; and the Triangles FLE, FEI, are similar, because the Angles FEL, FIE, are both Right Angles, and the Angle F is common: Therefore the Triangles FIE and KGE are similar, whence KG (or $\sqrt{rr-ss}$) : KE (or r) : : FI (or $\dot{s}$) : FE (or $\dot{a}$); that is, $\dot{a}=\frac{r\dot{s}}{\sqrt{rr-ss}}$.

Now $\sqrt{rr-ss}=r-\frac{ss}{2r}-\frac{s^4}{8r^3}-\frac{s^6}{16r^5}-\frac{s^8}{128r^7}$, &c. by extracting the Square Root:

And if $r\dot{s}$ be divided by that Series, the Quotient, $\dot{s}+\frac{s^2\dot{s}}{2rr}+\frac{3s^4\dot{s}}{8r^4}+\frac{5s^6\dot{s}}{16r^6}+\frac{35s^8\dot{s}}{128r^8}$, &c. will be the Fluxion of the Arc; therefore the Fluent thereof, *viz.* $\left(s+\frac{s^3}{2.3rr}+\frac{3s^5}{2.4.5r^5}+\frac{5s^7}{2.4.2.7r^6}+\frac{35s^9}{2.4.2.8.9r^8}, \text{\&c. or}\right)$ $s+\frac{s^3}{2.3r^2}+\frac{3.3s^5}{2.3.4.5r^4}+\frac{3.3.5.5s^7}{2.3.4.5.6.7r^6}+\frac{3.3.5.5.7.7s}{2.3.4.5.6.7.8.9r^8}$ &c.

&c. will be equal to the Arc of a Circle whose Radius is r, and Sine s. But if r be put equal to Unity, then $s + \frac{1}{2.3}s^3 + \frac{3.3}{2.3.4.5}s^5 + \frac{3.3.5.5}{2.3.4.5.6.7}s^7$, *&c.* will express the Length of the Arc a.

EXAMPLE.

Let it be required to find the Length of the Arc of 30 Degrees to 6 Places of Decimals, the Radius being Unity.

Here $s = \frac{1}{2}$, and $ss = \frac{1}{4}$; whence the Operation may be as follows:

s	= ,5000000		s	= ,5000000
s^3	= ,1250000		$\frac{s^3}{6}$	= 208333
s^5	= 312500		$\frac{3s^5}{40}$	= 23437
s^7	= 78125		$\frac{5s^7}{112}$	= 3487
s^9	= 19531		$\frac{35s^9}{1152}$	= 593
s^{11}	= 4882		$\frac{63s^{11}}{2816}$	= 109
s^{13}	= 1220		$\frac{231s^{13}}{13312}$	= 21
s^{15}	= 305		$\frac{143s^{15}}{10240}$	= 4
				,5235984

Hence the Length of the Arc of 30 Degrees is ,523598+. Now if this Arc be multiplied by 6, we shall have the Length of the Arc of the Semicircle in such Parts as the Radius is 1, or of the whole Circumference in such Parts as the Diameter is 1, *viz.* 3,14159+.

But there is no Series so easy to be retained in the Memory, and so readily put in Practice, for obtaining the Ratio of the Diameter of the Circle to its Circumference, as that which is derived from the Tangent. For if t be put equal to the Tangent of any Arc, then $a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9$, *&c.*

Now

Now the Radius being Unity the Sine of 30 Degrees $= \frac{1}{2}$, and conſequently the Coſine $= \sqrt{\frac{3}{4}}$; and becauſe the Coſine is to the Right Sine, as the Radius to the Tangent; it will be $\sqrt{\frac{3}{4}} : \sqrt{\frac{1}{4}} :: 1 : \sqrt{\frac{1}{3}}$, the Tangent of $30° \, 00' = t$, whence $tt = \frac{1}{3}$. Wherefore, if the Root of $\frac{1}{3}$ be divided continually by 3, and the ſeveral Quotients by the odd Numbers ſucceſſively, *viz.* the firſt by 3, the ſecond by 5, *&c.* the Sum of the affirmative Quotients made leſs by all the negative ones, will be the Arc of 30 Degrees.

And becauſe the Arc of 30 Degrees is $\frac{1}{6}$ Part of the Semicircumference, if inſtead of $\sqrt{\frac{1}{3}}$ be taken $6\sqrt{\frac{1}{3}} = \sqrt{12}$, we ſhall have the Semicircumference in ſuch Parts as the Radius is Unity; or the whole Circumference, the Diameter being Unity.

The OPERATION ſtands thus:

$\sqrt{12}$	$=3,464102$	$+\ 3,464102$	
3)	1,154701(		$-\ ,384900$
5)	,384900(	$+\ 76980$	
7)	,128300(		$-\ 18329$
9)	42767(	$+\ 4752$	
11)	14256(		$-\ 1296$
13)	4752(	$+\ 366$	
15)	1584(		$-\ 106$
17)	528(	$+\ 31$	
19)	176(		$-\ 9$
		$+\ 3,546231$	$-\ ,404640$

Whence $3,546231 - ,404640 = 3,141591$ the ſame as before. The Impoſſibility of expreſſing the exact Proportion of the Diameter of a Circle to its Circumference, by any received Way of Notation, has put the moſt celebrated Men, in all Ages, upon approximating the Truth as near as poſſible; there being a Neceſſity of a near Quadrature, inaſmuch as it is the Baſis upon which the moſt uſeful Branches of the Mathematics are built. And after the famous *Van Ceulen*, who carried it to 36 Places of Decimals (which he order'd to be engraven on his Tomb-ſtone, thinking he had ſet Bounds to farther Improvements), the firſt that attempted it with Succeſs was the moſt indefatigable Mr. *Abraham Sharp*, who, by a double Computation, *viz.* from the Sine of 6 Degrees one Way, and from the Sine and Coſine of 12 De-

12 Degrees another Way, carried it to twice the Number of Places that *Van Ceulen* had done, *viz.* 72.

And since that time Mr. *Machin*, late Professor of Astronomy in *Gresham College*, and Secretary to the *Royal Society*, by a different Method of Computation, has carried it to 100 Places, almost triple the Number that *Van Ceulen* had done, which not only confirms Mr. *Sharp's* Quadrature, but shews us, that, if the Diameter be 1,00000, &c. the Circumference will be 3,14159. 26535. 89793. 23846. 26433. 83279. 50288. 41971. 69399. 37510. 58209. 74944. 59230. 78164. 06286. 20899. 86280. 34825. 34211. 70680 + of the same Parts.

Which is a Degree of Exactness far surpassing all Imagination; being, by Estimation, more than sufficient to calculate the Number of Grains of Sand that may be comprehended within the Sphere of the Fixed Stars.

The late Mr. *Cunn's* Series for determining the Periphery of an Ellipsis (who was my Predecessor in the Mathematical School, erected by *Frederic Slare*, M. D. and established by a Decree of the High Court of Chancery for qualifying Boys for the Sea-Service) being new and curious, this Opportunity is taken of making in public.

Let A be equal to a Quadrant of the Circle circumscribing the Ellipsis, whose Periphery is required. Then $A \times$

$$1 - \frac{1.1}{2.2}e^2 - \frac{1.3.1}{2.4.8}e^4 - \frac{1.3.5.1}{2.4.6.16}e^6 - \frac{1.3.5.7.5}{2.4.6.8.128}e^8 - \frac{1.3.5.7.9.7}{2.4.6.8.10.256}e^{10} - \frac{1.3.5.7.9.11.21}{2.4.6.8.10.12.1024}e^{12},\ \&c.$$

is the Periphery of a Quadrant of the Ellipsis, where $ee = \frac{tt - cc}{tt}$, t being the Semi-transverse Diameter, and c the Semi-conjugate.

When this Series came to hand, it was imperfect, inasmuch as there were only the first five Terms without the Law of Continuation: But, being desirous of rendering it compleat, after some Consideration, I found the Law to be as follows: It is plain by Inspection, that the Numerators and Denominators of each Term are compos'd of Numbers that run in arithmetical Progression, except the last in each Term, *viz.* $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{5}{128}$, &c. and those being found by the continual Multiplication

tion of these Fractions, $\frac{1}{1}\times\frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}\times\frac{5}{8}\times\frac{7}{10}\times\frac{9}{12}$, &c. the Law of continuing the whole Series as above, is evident. Whence, by a well known Method of substituting Capital Letters for each *Term respectively, the following Series is deduced*, viz. $A\times 1-\frac{1}{4}ee-\frac{1.3}{4.4}e^2B-\frac{3.5}{6.6}e^2C-\frac{5.7}{8.8}e^2D-\frac{7.9}{10.10}e^2E-\frac{9.11}{12.12}e^2F$, &c. where the Law of Continuation is evident also, since each Capital Letter is equal to its precedent Term, *viz.* $B=\frac{1}{4}ee$, $C=\frac{1.3}{4.4}e^2B$, &c. and without doubt in Practice is preferable to the former Series: But the Investigation of that, on which this last depends, is omitted; purely on account of its being foreign to the present Subject.

But to return; if the Series expressing the Length of the Arc, *viz.* $s+\frac{1}{6}s^3+\frac{3}{40}s^5$, &c. be reversed, we shall have the Value of s in the Terms of a, and consequently a direct Method for finding the Sine of any Arc from its Length given. Thus,

If $a=s+\frac{1}{6}s^3+\frac{3}{40}s^5+\frac{5}{112}s^7$, &c.
Then $s=a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7$, &c.

$$\text{Or } s=a-\frac{a^3}{2.3}+\frac{a^5}{2.3.4.5}-\frac{a^7}{2.3.4.5.6.7},\ \&c.$$

$$\left.\begin{aligned}&\text{For put } s=Aa+Ba^3+Ca^5,\ \&c.\\&\text{Then } \tfrac{1}{6}s^3=\tfrac{1}{6}A^3a^3+\tfrac{1}{2}A^2Ba,\ \&c.\\&\text{And } \tfrac{3}{40}s^5=+\tfrac{3}{40}A^5a^5,\ \&c.\end{aligned}\right\}=a.$$

And consequently $A.a=a$; whence $A=1$:
Also, $B+\frac{1}{6}A^3=0$; or $B=(-\frac{1}{6}A^3=)-\frac{1}{6}$:
Again $C+\frac{1}{2}A^2B+\frac{3}{40}A^5=0$; or $C=-\frac{1}{2}A^2B-\frac{3}{40}A^5$:
That is $C=(\frac{1}{2}\times 1\times\frac{1}{6}-\frac{3}{40}=\frac{10}{120}-\frac{9}{120}=)\frac{1}{120}$
Wherefore $A=1$, $B=-\frac{1}{6}$, $C=\frac{1}{120}$, &c. and consequently, $s=a-\frac{a^3}{6}+\frac{a^5}{120}$, &c. From which three Terms the Law of Continuation is easily discovered: Therefore $s=a-\frac{a^3}{2.3}+\frac{a^5}{2.3.4.5}-\frac{a^7}{2.3.4.5.6.7}+\frac{a^9}{2.3.4.5.6.7.8.9}$, &c.

Whence,

Whence, substituting A for a, and we shall have $A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9}$ &c. for the *Newtonian* Series, according to our Author's Form, for finding the Sine of any Arc, its Length being given. Q. E. I.

Again, because the Square of the Radius, made less by the Square of the Sine, is equal to the Square of the Cosine, by the second Proposition of our Author's Elements of Plane Trigonometry; it follows, that if from the Square of the Radius $= 1$ be taken the Square of the Sine $= a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the Square Root of the Remainder will be the Cosine $= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6$, &c. Thus,

$$\begin{array}{l} s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\ s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\ \hline a^2 - \frac{1}{6}a^4 + \frac{1}{120} \\ \qquad - \frac{1}{6}a^4 + \frac{1}{36}a^6, \text{ \&c.} \\ \qquad\qquad + \frac{1}{120}a^6, \text{ \&c.} \\ \hline ss = a^2 - \frac{1}{3}a^4 + \frac{2}{45}a^6, \text{ \&c.} \end{array}$$

which, being taken from the Squares of the Radius 1, leaves $1 - aa + \frac{1}{3}a^4 - \frac{2}{45}a^6$, &c. the Square Root of which will be the Cosine.

$$1 - aa + \tfrac{1}{3}a^4 - \tfrac{2}{45}a^6, \text{ \&c.} \left(1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \tfrac{1}{720}a^6, \text{ \&c.} \right.$$

$$\begin{array}{rl} 2 - \frac{1}{2}aa) & -aa + \frac{1}{3}a^4 \\ & -aa + \frac{1}{4}a^4 \\ \hline 2 - aa, \text{ \&c.}) & \frac{1}{12}a^4 - \frac{2}{45}a^6, \text{ \&c.} \\ & \frac{1}{12}a^4 - \frac{1}{24}a^6, \text{ \&c.} \\ \hline 2 - aa, \text{ \&c.}) & -\frac{1}{360}a^6, \text{ \&c.} \\ & -\frac{1}{720}a^6, \text{ \&c.} \end{array}$$

Wherefore putting A for a, we shall have the Cosine $1 - \frac{A^2}{2} + \frac{A^4}{24} - \frac{A^6}{720}$, &c. or $1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8}$, &c. according to the Author's Form. Q. E. I.

But because those Series, as our Author observes, converge very slowly, especially when the Arc is nearly equal

equal to the Radius; he therefore devifed (Page 2; 8.) other Series, whofe Inveftigation may be as follows:

Let the Arc, whofe Sine is fought, be the Sum or Difference of two Arcs, *viz.* A+z, or A—z; and let the Sine of the Arc A be called *a*, and the Cofine *b*.

Now, if the Arc DF=DE, *Prop. 5th of the Elements of Trigonometry*, be called z, then its Sine FO will, by the *Newtonian* Series, be $= z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7}$, &c. and its Cofine CO $= 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6}$, &c. and becaufe CD : DK : : CO : OP; therefore OP $= a - \frac{az^2}{1.2} + \frac{az^4}{1.2.3.4} - \frac{az^6}{1.2.3.4.5.6}$, &c.

Again, becaufe the Triangles CDK, FOM, are fimilar, it will be, as CD : CK : : FO : FM; whence FM $= \frac{bz}{1} - \frac{bz^3}{1.2.3} + \frac{bz^5}{1.2.3.4.5} - \frac{bz^7}{1.2.3.4.5.6.7}$, &c.

But OP + FM + IF, the Sine of the Arc BF, *viz.* A + z; confequently the Sum of thofe Series, *viz.* $a + \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5}$, &c. is the Sine of the Arc A+z. And becaufe FM = MG, therefore their Difference $a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5}$, &c. is the Sine of the Arc A—z, *viz.* EL.

And again, becaufe CD : CK : : CO : CP; therefore CP $= b - \frac{bz^2}{1.2} + \frac{bz^4}{1.2.3.4} - \frac{bz^6}{1.2.3.4.5.6} + \frac{bz^8}{1.2.3.4.5.6.7.8}$, &c. and by reafon of the fimilar Triangles CDK, FMO, it will be, as CD : DK : : FO : MO. Whence MO $= az - \frac{az^3}{1.2.3} + \frac{az^5}{1.2.3.4.5} - \frac{az^7}{1.2.3.4.5.6.7}$, &c.

But

But CP—MO=CI, the Cosine of the Arc A+z. Wherefore the Cosine of the Arc A + z is $b - \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} - \frac{}{1.2.3.4.5.6}$ &c.

And because MO = PL therefore CP + MO = CL, and consequently the Cosine of the Arc A—z= $+\frac{az}{1} - \frac{bz^2}{1.2} - \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} \cdot \frac{az^5}{1.2.3.4.5}$ &c. Q. E. I.

Now the Arc A is an Arithmetical Mean between the Arcs A—z and A+z, and the Difference of their Sines are $\frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6}$, &c. $\frac{bz}{1} + \frac{az}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6}$, &c. Whence the Difference of the Differences, or second Difference, is $\frac{2az^2}{1.2} - . \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6}$, &c. or $2a \times \overline{\frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6}}$, &c. Which Series is equal to double the Sine of the mean Arc, drawn into the versed Sine of the Arc z, and converges very soon; so that if z be the Arc of the first Minute of the Quadrant, our Author says the first Term of the Series gives the second Difference to 15 Places of Figures, and the second Term to 25 Places.

Whence the following Rule is derived for finding the Sine of the Arc A+z, or A—z.

RULE.

From double the Sine of the mean or middle Arc, subtract the second Difference found by the Theorem, and from the Remainder subtract the Sine of the given Extreme, whether it be the greater or least; and the Remainder will be the Sine of the other Extreme.

EXAMPLE.

Let it be required to find the Sine of 30° 01′; the Sines of 30° 00′, and 29° 59′, being both given.

Here

Here 30° 00′ is the mean Arc, whose Sine is ,500000 00000; and the Sine of 29° 59′, the given Extreme, is ,49974806226, and the Length of the Arc z, *viz.* one Minute, is ,000290888208; which squared and multiplied by the Sine of the mean Arc 50000, *&c.* according to the Direction of the Theorem, the Product will be the second Difference, equal to ,000000042307; which subtracted from double the Sine of the mean Arc, equal to 1, the Remainder will be ,999999957693; from which subtract the Sine of the given Extreme (which in this Case is the least), and there will remain ,50025189543 for the Sine of 30° 01′, the greater Extreme.

This Method of making the Sines, however it may appear at first Sight, is so far from being tedious or troublesome, that I look upon it to be the most eligible of any other whatsoever: For the Square of z being once determined, and the several Multiples of it by the nine Digits made, and set down in a Table orderly, all the Sines may be made by Addition and Subtraction only; as indeed our Author hints they may by the Method demonstrated in the tenth Proposition of the Elements of Trigonometry; but this is evidently preferable to that, tho' a good Method too; and by which all the Sines of the Quadrant, I presume, were wont to be made, at least as far as 30, or 60 Degrees; for after the Sines as far as 60 Degrees are obtained, all the others may be had by Addition only; and notwithstanding there are other excellent Theorems, which contribute very much towards finishing and confirming the Truth of the whole Canon; yet this deduced from our Author's Series, I deem the most elegant and fit for Practice, because the Difference of the Differences of the Sines being what is always required to be found, there will be seven Cyphers, at least, before the significant Figures of the said Difference; which is the Product made by the Square of z, into the Sine of the mean Arc: So that to have the Sine true to ten Places, there will not be occasion to find above four or five Figures in the Product, which, according to the common Method of contracted Multiplication, may be obtained with very few Figures. Thus, for Instance, the Sine of 30° 02′ may be had to ten Places by a wonderful easy Opeartion; the Sines of 30° 01′ and 30° 00′ being both given.

EX-

EXAMPLE.

The Sine of 30° 01′ is	,50025189543
The Square of z inverted	26480000000

40020
2001
400
10

42331

Whence the Product is ,00000004233I true to eleven Places at least. Wherefore if, according to the Rule, from double the Sine of the middle Arc = 1,00050379086 we subtract the said Product, ,00000004233

And from the Remainder	1,00050374853
the Sine of 30° 00′ the given Extreme	,50000000000
be subtracted	,50050374853

There will remain ,50050374853 for the Sine of 30° 02′ the other Extreme; than which, nothing of this Nature can be desired more easy.

SCHOLIUM.

Because the Difference of the Differences of the Sines, or second Difference, has always 7 Cyphers before the significant Figures; it follows, that the whole Canon, where the Sines consist but of 6 Places, which is as far as our Tables for common Practice need extend, may be performed chiefly by Addition and Subtraction only, without forming Multiples of the Square of z by the nine Digits; tho' perhaps it may be necessary to use the Method of contracted Multiplication every 5th Minute, to confirm the Truth, lest, in continual doubling and subtracting, an Error should arise in the Right-hand Figure: However, as it may be safely used for 5 Minutes together, and sometimes more, it will render the Whole very easy.

Note, The Square of z in this Case, *viz.* the Arc of 5 Minutes, is ,000002115.

 Thus

Thus having investigated the *Newtonian* and our Author's Series, and exemplified the latter, by making the Sines of 30° 01′ and 30° 02′, and withal shewn how, from the Sine of the Arc given, to find the Length of that Arc, and consequently the Circumference of the whole Circle; I shall beg Leave, before I treat of the Construction of Logarithms, to shew how, from the known Ratio of the Diameter to the Circumference, or any other Ratio whatsoever, that a Set of integral Numbers may be found, whose Ratios shall be the nearest possible to the Ratio given; for which I hope to be excused, and the rather, because I believe this Method of determining them was never before published.

RULE.

Divide the Consequent by the Antecedent, and the Divisor by the Remainder, and the last Divisor by the last Remainder, and so on till nothing remains.

Then for the Terms of the first Ratio, Unity will always be the Antecedent, and the first Quotient the first Consequent.

For the Terms of the second Ratio.

Multiply the last $\left\{\begin{array}{l}\text{Antecedent}\\ \text{Consequent}\end{array}\right\}$ by the 2d Quotient; and to the Product add $\left\{\begin{array}{l}\text{Nothing;}\\ \text{Unity;}\end{array}\right\}$ and so will the Result be the second $\left\{\begin{array}{l}\text{Antecedent.}\\ \text{Consequent.}\end{array}\right.$

For all the following Ratios:

Multiply the last $\left\{\begin{array}{l}\text{Antecedent}\\ \text{Consequent}\end{array}\right\}$ by the next Quotient, and to the Product add the last $\left\{\begin{array}{l}\text{Antecedent}\\ \text{Consequent}\end{array}\right\}$ but one; and so will the Sum be the present $\left\{\begin{array}{l}\text{Antecedent.}\\ \text{Consequent.}\end{array}\right.$

EXAMPLE.

Let it be required to find a Rank of Ratios, whose Terms are integral, and the nearest possible to the following Ratio, *viz.* of 10000 to 31416, which expresses nearly the Proportion of the Diameter of the Circle to its Circumference.

But because the Terms of the Ratio are not prime to each other, they must therefore be reduced to their least Terms.

Whence $\frac{10000}{31416}=\frac{1250}{3927}$, and then 3927 divided by 1250, and 1250 by the Remainder, &c. will be as follows:

```
1250 ) 3927 ( 3
        177 ) 1250 ( 7
                11 ) 177 ( 16
                        1 ) 11 ( 11
```

So the first Antecedent is 1, and the first Consequent 3.

$$\text{Last}\left\{\begin{array}{ll}\text{Anteced.} & 1\\ \text{Conseq.} & 3\end{array}\right\}\times 7=\left\{\begin{array}{l}7\\21\end{array}\right\}\text{And}\left\{\begin{array}{l}7+0=7 \text{ the second Antec.}\\ 21+1=22 \text{ the second Conf.}\end{array}\right.$$

Which 7 and 22 is *Archimedes*'s Proportion.

$$\text{Last}\left\{\begin{array}{ll}\text{Anteced.} & 7\\ \text{Conseq.} & 22\end{array}\right\}\times 16=\left\{\begin{array}{l}112\\352\end{array}\right\}\text{And}\left\{\begin{array}{l}112+1=113 \text{ the 3d Ant.}\\ 352+3=355 \text{ the 3d Conf.}\end{array}\right.$$

Which Terms 113 and 355 is *Metius*'s Proportion.

$$\text{Last}\left\{\begin{array}{ll}\text{Antecedent} & 113\\ \text{Consequent} & 355\end{array}\right\}\times 11=\left\{\begin{array}{l}1243\\3905\end{array}\right\}\text{And}\left\{\begin{array}{l}1243+7=1250\\ 3905+22=3927.\end{array}\right.$$

Producing the same Antecedent and Consequent as at first; which, as it is ever the Property of the Rule so to do, proves, at the same Time, that no Error has been committed thro' the whole Operation.

$$\text{Whence, as } 1250:3927::\left\{\begin{array}{r}1:3\\7:22\\113:355\end{array}\right\}\begin{array}{l}\text{For the}\\ \text{Terms}\\ \text{of the}\end{array}\left\{\begin{array}{l}1\\2\\3\end{array}\right\}\text{Ratio.}$$

But it must be observed, that 1 to 3 does not express the Ratio so near as 7 to 22; nor 7 to 22 so near as 113 to

to 355; that is, the larger the Terms of the Ratio are, the nearer they approach the Ratio given.

Mr. *Molyneux*, in his Treatise of Dioptrics, informs us, that when Sir *Isaac Newton* set about, by Experiments, to determine the Ratio of the Angle of Incidence, to the refracted Angle, by the means of their respective Sines; he found it to be, from Air to Glass, as 300 to 193, or, in the least round Numbers, as 14 to 9. Now, if it be as 300 is to 193, it will readily appear, by the Rule, whether they are such integral Numbers, whose Ratio is the nearest possible to the given Ratio.

```
193 ) 300 ( 1
      107 ) 193 ( 1
             86 ) 107 ( 1
                   21   86   4
                         2   21   10
                              1    2 (
```

For, dividing the great Number by the less, and the less by the Remainder, &c. the Operation will shew that the Numbers 193 and 300 are prime to each other; and that the first Antecedent is 1, as also the first Consequent.

$$\text{Whence}\begin{Bmatrix}1\\1\end{Bmatrix}\times 1=\left\{\begin{matrix}1\\1\end{matrix}\right.\ \text{And}\left\{\begin{matrix}1+0=1\text{ the 2d Ant.}\\1+1=2\text{ the 2d Con.}\end{matrix}\right.$$

$$\text{Again}\begin{Bmatrix}1\\2\end{Bmatrix}\times 1=\left\{\begin{matrix}1\\1\end{matrix}\right.\ \text{And}\left\{\begin{matrix}1+1=2\text{ the 3d Ant.}\\1+2=3\text{ the 3d Con.}\end{matrix}\right.$$

$$\text{Again}\begin{Bmatrix}2\\3\end{Bmatrix}\times 1=\left\{\begin{matrix}1\\12\end{matrix}\right.\ \text{And}\left\{\begin{matrix}8+1=9\text{ the 4th Ant.}\\12+2=14\text{ the 4th Con.}\end{matrix}\right.$$

Hence, the fourth Antecedent and Consequent make the Ratio to be as 9 to 14, or, inversly, as 14 to 9; which not only agrees with Mr. *Molyneux*, but at the same Time discovers, that they are nearer to the given Ratio, than any other integral Numbers less than 92 and 143; which are the nearest of all to the given Ratio, as will appear by repeating the Process, according to the Direction of the Rule.

Sir *Isaac Newton* himself determines the Ratio out of Air into Glass to be as 17 to 11; but then he speaks of the red Light. For that great Philosopher, in his Differ-

Differtations concerning Light and Colours, publiſhed in the *Philoſophical Tranſactions*, has at large demonſtrated, as alſo in his *Optics*, that the Rays of Light are not all homogeneous, or of the ſame Sort, but of different Forms and Figures, ſo that ſome are more refracted than others, tho' they have the ſame or equal Inclinations on the Glaſs: Whence there can be no conſtant Proportion ſettled between the Sines of the Angles of Incidence, and of the refracted Angles.

But the Proportion that comes neareſt Truth, for the middle or green-making Rays of Light, it ſeems, is nearly as 300 to 193, or 14 to 9. In Light of other Colours the Sines have other Proportions. But the Difference is ſo little, that it need ſeldom to be regarded, and either of thoſe mentioned for the moſt Part is ſufficient for Practice. However, I muſt obſerve, that the Notice here taken either of the one or the other, is more to illuſtrate the Rule, and ſhew, as Occaſion requires, how to expreſs any given Ratio in ſmaller Terms, and the neareſt poſſible, with more Eaſe and Certainty, than any Deſign in the leaſt of touching upon Optics.

Wherefore, leſt this ſmall Digreſſion from the Subject in hand, and indeed even from my firſt Intentions, ſhould tire the Reader's Patience, I ſhall not preſume more, but immediately proceed to the Conſtruction of Logarithms.

Of the Conſtruction of Logarithms.

THE Nature of which, tho' our Author has ſufficiently explained in the Deſcription of the Logarithmical Curve; yet, before we attempt their Conſtruction, it will be neceſſary to premiſe:

That the Logarithm of any Number is the Exponent or Value of the Ratio of Unity to that Number; wherein we conſider Ratio, quite different from that laid down in the fifth Definition of the 5th Book of theſe Elements; for, beginning with the Ratio of Equality, we ſay 1 to 1 = 0; whereas, according to the ſaid Definition, the Ratio of 1 to 1 = 1; and conſequently the Ratio here mentioned is of a peculiar Nature, being affirmative when increaſing, as of Unity to a greater Number; but negative when decreaſing. And

as the Value of the Ratio of Unity to any Number is the Logarithm of the Ratio of Unity to that Number, so each Ratio is supposed to be measured by the Number of equal Ratiunculæ contained between the two Terms thereof: Whence, if in a continued Scale of mean Proportionals, infinite in Number, there be assumed an infinite Number of such Ratiunculæ, between any two Terms in the same Scale; then that infinite Number of Ratiunculæ is to another infinite Number of the like and equal Ratiunculæ between any other two Terms, as the Logarithm of the one Ratio is to the Logarithm of the other.

But if, instead of supposing the Logarithms composed of a Number of equal Ratiunculæ proportionable to each Ratio, we shall take the Ratio of Unity to any Number to consist always of the same infinite Number of Ratiunculæ, their Magnitudes in this Case will be as their Number in the former. Wherefore, if between Unity, and any two Numbers proposed, there be taken any Infinity of mean Proportionals; the infinitely little Augments or Decrements of the first of those Means in each from Unity will be Ratiunculæ; that is, they will be the Fluxions of the Ratio of Unity to the said Numbers; and because the Number of Ratiunculæ in both are equal, their respective Sums, or whole Ratios, will be to each other as their Moments or Fluxions; that is, the Logarithm of each Ratio will be as the Fluxion thereof. Consequently, if the Root of any infinite Power be extracted out of any Number, the Difference of the said Root from Unity shall be as the Logarithm of that Number. So that Logarithms, thus produced, may be of as many Forms as we please to assume infinite Indices of the Power whose Root we seek. As, if the Index be supposed 100000, &c. we shall have the Logarithms invented by *Neper*; but if the said Index be 230258, &c. those of Mr. *Briggs* will be produced.

Wherefore, if $1+x$ be any Number whatsoever, and n infinite, then its Logarithm will be as $\overline{1+x}^{\frac{1}{n}} - 1 = \frac{1}{n} \times x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, &c. For the infinite Root of $1+x$ without its Unciæ or prefixed Numbers,

bers, is $1+x+xx+xxx+xxxx$, &c. and the celebrated binomial Theorem invented by Sir *Isaac Newton* for determining them is $1\times\frac{1}{n}\times\frac{\frac{1}{n}-1}{2}\times\frac{\frac{1}{n}-2}{3}\times\frac{\frac{1}{n}-3}{4}$ &c. or in this Case rather $1\times\frac{1}{n}\times\frac{-1}{2}\times\frac{-2}{3}\times\frac{-3}{4}\times\frac{-4}{5}$ &c. for $\frac{1}{n}$ being an Infinitesimal, is rejected; whence the infinite Root of $\overline{1+x}^{\frac{1}{n}}=1+\frac{x}{n}-\frac{x^2}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. and the Excess thereof above Unity, *viz.* $\frac{x}{n}-\frac{x^2}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. is the Augment of the first of the mean Proportionals between Unity and $1+x$, which therefore will be as the Logarithm of the Ratio of 1 to $1+x$, or as the Logarithm of $1+x$. But as $\overline{1+x}^{\frac{1}{n}}-1$ is a Ratiuncula, it must be multiplied by 10000, &c. infinitely, which will reduce it to Terms fit for Practice, making the Logarithm of the Ratio of 1 to $1+x=\frac{1000,\ \&c.}{n}\times\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$, &c. whence if the Index n be taken 1000, &c. as in *Neper*'s Form, the Logarithms will be simply $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$, &c. But as n may be taken at Pleasure, the several Scales of Logarithms to such Indices will be as $\frac{1000,\ \&c.}{n}$, or reciprocally as their Indices.

Again, if the Logarithm of a decreasing Ratio be sought, the infinite Root of $1-x=\overline{1-x}^{\frac{1}{n}}$ will be found by the like Method to be $1-\frac{x}{n}-\frac{x^2}{2n}-\frac{x^3}{3n}-\frac{x^4}{4n}$, &c. which subtract from Unity, and the Decrement of the first of the infinite Number of Proportionals will appear to be $\frac{1}{n}\times\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$, &c. which

which expresses the Logarithm of the Ratio of 1 to $1-x$, or the Logarithm of $1-x$, according to *Neper's* Form, if the Index n be put $=10000$, *&c.* as before.

And to find the Logarithm of the Ratio of any two Terms, a the lesser, and b the greater, it will be as $a : b :: 1 : 1+x$; whence $1+x=\frac{b}{a}$; and $x=\left(\frac{b-a}{a}\text{, or}\right)$ the Difference divided by the lesser Term when 'tis an increasing Ratio, and $\frac{b-a}{b}$ when 'tis decreasing.

Wherefore, putting $d=$Difference between the two Terms a and b, the Logarithms of the same Ratio may be doubly expressed, and accordingly is either

$$\frac{1}{n}\times\overline{\frac{d}{a}-\frac{d^2}{2a^2}+\frac{d^3}{3a^3}-\frac{d^4}{4a^4}},\ \&c. \text{ or}$$

$$\frac{1}{n}\times\overline{\frac{d}{b}+\frac{d^2}{2b^2}+\frac{d^3}{3b^3}+\frac{d^4}{4b^4}},\ \&c.$$ both producing the same Thing.

But if the Ratio of a to b be supposed to be divided into two Parts, *viz.* into the Ratio of a to the arithmetical Mean between the two Terms, and the Ratio of the said arithmetical Mean to the other Term b, then will the Sum of the Logarithms of those two Ratios be the Logarithm of the Ratio of a to b. Wherefore substituting $\frac{1}{2}s$ for $\frac{1}{2}a+\frac{1}{2}b$, and it will be, $\frac{1}{2}s : a :: 1 : 1-x$; whence $x=\frac{\frac{1}{2}s-a}{\frac{1}{2}s}=\left(\frac{s-2a}{s}=\right)\frac{d}{s}$; and again, as $\frac{1}{2}s : b :: 1 : 1+x$; $x=\frac{b-\frac{1}{2}s}{\frac{1}{2}s}=\left(\frac{2b-s}{s}=\right)\frac{d}{s}$: Therefore substituting $\frac{d}{s}$, for x, we shall have the Logarithms of those Ratios; *viz.*

$$\frac{1}{n}\times\overline{\frac{d}{s}+\frac{d^2}{2s^2}+\frac{d^3}{3s^3}+\frac{d^4}{4s^4}},\ \&c. \text{ and}$$

$$\frac{1}{n}\times\overline{\frac{d}{s}-\frac{d^2}{2s^2}+\frac{d^3}{3s^3}-\frac{d^4}{4s^4}},\ \&c.$$

The Sum of which two Logarithms, *viz.*

$$\frac{1}{n}\times 2\times\overline{\frac{d}{s}+\frac{d^3}{3s^3}+\frac{d^5}{5s^5}+\frac{d^7}{7s^7}},\ \&c.$$ is the

the Logarithm of the Ratio of a to b, whose Difference is d, and Sums s; which Series, without the Index n, is, by-the-bye the Fluent of the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, assuming d, to be the flowing Quantity, for the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, is $\frac{2s\dot{d}}{ss-dd}=2\times\overline{\frac{\dot{d}}{s}+\frac{d^2\dot{d}}{s^3}+\frac{d^4\dot{d}}{s^5}+\frac{d^6\dot{d}}{s^7}+}$, &c. whose Fluent $2\times\overline{\frac{d}{s}+\frac{d^3}{3s^3}+\frac{d^5}{5s^5}+\frac{d^7}{7s^7}}$, is *Neper's* Logarithm of $\frac{s+d}{s-d}$, and the same as above, abating the Index n. This Series, either Way obtained, converges twice as swift as the former, and consequently is more proper for the Practice of making Logarithms: Thus put $a=1$, and b any Number at Pleasure; then $\frac{d}{s}=\frac{b-1}{b+1}$, which assume $=e$, and then $b=\frac{1+e}{1-e}$; and because $\frac{d}{s}=e$, therefore have we for

THEOREM I.

The Log. of $b=\left(\frac{1+e}{1-e}=\right)\frac{2}{n}\times\overline{e+\frac{1}{3}e^3+\frac{1}{5}e^5,\ \&c.}$

To illustrate this Theorem: Let it be required to find the Logarithm of 2 true to 7 Places.

Note, That the Index must be assumed of a Figure or two more than the intended Logarithm is to have.

EXAMPLE.

Here $(b=)\frac{1+e}{1-e}=2$; therefore $1+e=(\overline{1-e}\times 2=)$ $2-2e$; and $3e=(2-1=)\ 1$; whence $e=\frac{1}{3}$, and $ee=\frac{1}{9}$.

The

The OPERATION [illegible] ftands thus:

$\frac{1}{3} = e$	$=$	,33333333	e	$=$	,3333[illegible]
e^3	$=$	3703704	[illegible]	$=$	[illegible]
e^5	$=$	411523	$\frac{1}{5}e^5$	$=$	8[illegible]
e^7	$=$	45725	$\frac{1}{7}e^7$	$=$	6[illegible]
e^9	$=$	508[illegible]	$\frac{1}{9}e^9$	$=$	[illegible]
e^{11}	$=$	565	$\frac{1}{11}e^{11}$	$=$	51
e^{13}	$=$	63	$\frac{1}{13}e^{13}$	$=$	5
					,34657359
					2

Whence *Neper*'s Logarithm of 2 is ,69314718

But ,69314718, multiplied by 3, will give 2,07944154 for the Logarithm of 8, inafmuch as 8 is the Cube or third Power of 2; and the Logarithm of 8 + Log. of $1\frac{1}{4}$ is equal to the Logarithm of 10, becaufe $8 \times 1\frac{1}{4} = 10$; wherefore to find the Logarithm of $1\frac{1}{4}$ we have $b = \frac{1+e}{1-e} = 1\frac{1}{4} = \frac{5}{4}$; whence $e = \frac{1}{9}$, and $ee = \frac{1}{81}$.

The OPERATION ftands thus:

$= e$	$=$	,11111111	e	$=$	,11111111
e^3	$=$	137174	$\frac{1}{3}e^3$	$=$	45725
e^5	$=$	1693	$\frac{1}{5}e^5$	$=$	339
e^7	$=$	21	$\frac{1}{7}e^7$	$=$	3
					,11157178

Whence *Neper*'s Logarithm of $1\frac{1}{4}$ is	,22314356
To which add the Logarithm of 8,	2,07944154
The Sum, *viz.*	2,30258510

is *Neper*'s Logarithm of 10. But if the Logarithm of 10 be made 1,000000, &c. as it is for Conveniency done in moft of the Tables extant, then $\frac{2302585}{n} = 1,000$, &c. Whence $n = 2302585$, &c. is the Index for *Briggs*'s Scale of Logarithms; and, if the above Work had been carried on to Places fufficient, the Index n would have been 2,30258, 50929, 94045, 68401, 79914,

79514, &c. and its Reciprocal, *viz.* $\frac{1}{n} = 0{,}43429,$ 44819, 03251, 82765, 11289, &c. which, by the Way, is the Subtangent of the Curve expreſſing *Briggs's* Logarithms; from the Double of which the ſaid Logarithms may be had directly.

For, becauſe $\frac{1}{n} = 0{,}4342944$, &c. $\therefore \frac{2}{n} =$, ... 3638, &c. which put $= m$, and then the Logarithm of $b = \frac{1+e}{1-e} = me + \frac{me^3}{3} + \frac{me^5}{5} + \frac{me^7}{7} + \frac{me^9}{9}$, &c.

EXAMPLE.

Let it be required to find *Briggs's* Logarithm of 2. Here $b = \frac{1+e}{1-e} = 2 \therefore e = \frac{1}{3}$, and $ee = \frac{1}{9}$.

The OPERATION ſtands thus:

m	$=$,868588963				
me	$=$,289529655		me	$=$	,289529655
me^3	$=$ 32169962	$\frac{1}{3}$	me^3	$=$	0723321
me^5	$=$ 3574440	$\frac{1}{5}$	me^5	$=$	714888
me^7	$=$ 397160	$\frac{1}{7}$	me^7	$=$	56738
me^9	$=$ 44129	$\frac{1}{9}$	me^9	$=$	4903
me^{11}	$=$ 4903	$\frac{1}{11}$	me^{11}	$=$	446
me^{13}	$=$ 545	$\frac{1}{13}$	me^{13}	$=$	42

Whence *Briggs's* Logarithm of 2 is 0,301029993

AGAIN:

Let it be required to find *Briggs's* Logarithm of 3. Now becauſe the Logarithm of 3 is equal to the Logarithm of 2 plus the Logarithm of $1\frac{1}{2}$ (for $2 \times 1\frac{1}{2} = 3$), therefore find the Logarithm of $1\frac{1}{2}$, and add it to the Logarithm of 2 already found, the Sum will be the Logarithm of 3, which is better than finding the Logarithm of 3 by the Theorem directly, inaſmuch as it will not converge ſo faſt as the Logarithm of $1\frac{1}{2}$; for the ſmaller the Fraction repreſented by e, which is deduced

deduced from the Number whose Logarithm is so the swifter does the Series converge.

Here $b = \frac{1+e}{} = \frac{3}{2}$ ∵ $2e \cancel{+} 2 = 3$ —

and $ee =$

The OPERATION is as follows:

m	$= ,868588963$		
me	$= ,173717792$	me	$= ,173717792$
me	$= 6948712$		$= ,0\ 2316237$
me	$= 277948$		$= 55590$
me'	$= 11118$	me'	$= 1588$
me^9	$= 445$	me^9	$= 50$
me^{11}	$= 18$	me^{11}	$= 2$

Briggs's Logarithm of $1\frac{1}{2}=$,176091259
To which add the Logarithm of $2=$,301029993
The Sum is the Logarithm of $3=$ 0,477121252

Again, to find the Logarithm of 4, because $2\times2=4$, therefore the Logarithm of 2 added to itself, or multiplied by 2, the Product 0,602059986 is the Logarithm of 4.

To find the Logarithm of 5, because $\frac{10}{2}=5$, therefore from the Logarithm of 10 1,000000000
subtract the Logarithm of 2 ,301029993
There remains the Logarithm of $5=$,698970007

And because $2\times3=6$; therefore
To find the Logarithm of 6,
To the Logarithm of 3 ,477121252
Add the Logarithm of 2 ,301029993
The Sum will be the Logarithm of $6=$,778151245

Which being known, the Logarithm of 7, the next prime Number, may be easily found by the Theorem; for because $6\times\frac{7}{6}=7$, therefore to the Logarithm of 6 add the Logarithm of $\frac{7}{6}$, and the Sum will be the Logarithm of 7.

EXAMPLE.

Here $b = \frac{1+e}{1-e} = \frac{7}{6}$ ∵ $e = \frac{1}{13}$, and $ee = \frac{1}{169}$.

$m =$

m	$=$	,868588963			
me	$=$	,066814536	me	$=$	,066814535
me^3	$=$	395352	$\frac{1}{3}\, me^3$	$=$	131784
me^5	$=$	2339	$\frac{1}{5}\, me^5$	$=$	468
me^7	$=$	1[illegible]	$\frac{1}{7}\, me^7$	$=$	2

Briggs's Logarithm of $\frac{7}{6}$	,066946789
To which add the Log. of 6	,778151245
The Sum is the Log. of 7 $=$	,845098034

Again, because $4 \times 2 = 8$; therefore

To the Logarithm of 4	,60205998
Add the Logarithm of 2	,30102999
The Sum is the Logarithm of 8	,90308997

And because $3 \times 3 = 9$; therefore

To the Logarithm of 3	,47712125
Add the Logarithm of 3	,47712125
The Sum is the Logarithm of 9 $=$	,95424250

And the Logarithm of 10 having been determined to be 1,00000000, we have therefore obtained the Logarithms of the first ten Numbers.

After the same manner the whole Table may be constructed; and as the prime Numbers increase, so fewer Terms of the Theorem are required to form their Logarithms; for in the common Tables, which extend but to seven Places, the first Term is sufficient to produce the Logarithm of 101, which is composed of the Sum of the Logarithms of 100 and $\frac{101}{100}$, because 100 $\times \frac{101}{100} = 101$, in which Case $b = \frac{1+e}{1-e} = \frac{101}{100}$ $\therefore e = \frac{1}{201}$; whence, in making of Logarithms according to the preceding Method, it may be observed, that the Sum and Difference of the Numerator and Denominator of the Fraction whose Logarithm is sought, is ever equal to the Numerator and Denominator of the Fraction represented by e; that is, the Sum of the Denominator, and the Difference, which is always Unity, is the Numerator; consequently, the Logarithm of any prime Number may be readily had by the Theorem, having the Logarithm either next above or below given.

Tho'

Tho' if the Logarithms next above and below that Prime are both given, then its Logarithm will be obtained somewhat easier. For by [illegible] the Difference of the Ratios which constitutes the first Theorem, [illegible]

$$m \times \frac{dd}{2ss} + \frac{d^4}{4s^4} + \frac{d^6}{6s^6} + \frac{d^8}{8s^8}, \text{ \&c. is the Logarithm}$$

of the Ratio of the arithmetical Mean to the geometrical Mean, which being added to the half Sum of the Logarithms, next above and below the Prime sought, will give the Logarithm of that prime Number, which for Distinction-sake, may be called *Theorem the second*, and is of good Dispatch, as will appear hereafter by an Example.

But the best for this Purpose is the following one, which is likewise derived from the same Ratios as Theorem the first. For the Difference of the Terms between ab and $\frac{1}{4}ss$, or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, is $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb = \overline{\frac{1}{2}a - \frac{1}{2}b}^2 = \frac{1}{4}dd = 1$, and the Sum of the Terms ab and $\frac{1}{4}ss$ being put $= y$, therefore (since y in this Case $= s$, and $d = 1$) it follows, that

$$\frac{1}{n} \times \frac{2}{y} + \frac{2}{3y^3} + \frac{2}{5y^5} + \frac{2}{7y^7}, \text{ \&c. is the Logarithm}$$

of the Ratio of ab to $\frac{1}{4}ss$: Whence

$$\frac{1}{n} \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}, \text{ \&c. is the Loga-}$$

rithm of the Ratio of $\sqrt{ab}$ to $\frac{1}{2}s$, which converges exceeding quick, and is of excellent Use for finding the Logarithms of prime Numbers, having the Logarithms of the Numbers next above and below given, as in Theorem the second.

EXAMPLE.

Let it be required to find the Logarithm of the prime Number 101; then $a = 100$, and $b = 102$; whence $y = 20401$; put $\frac{1}{n} = m = ,4342944819$, &c. Then the Series will stand thus, $\frac{m}{y} + \frac{m}{3y^3} + \frac{m}{5y^5} + \frac{m}{7y^7}$, &c.

And

And $m = ,43429$, &c. divided by $y = 20401$, quotes	,0000212879017
Therefore to the half Sum of the Logarithms of 100 and 102 =	2,004380085881 0
Add the said Quote	0,0000212879017
And the Sum, viz.	2,0043213737827

is the Logarithm of 101 true to 12 Places of Figures, and obtained by the first Term of the Series only; whence it is easy to perceive what a vast Advantage the second Term would have, were it put in Practice, since m is to be divided by 3 multiplied into the Cube of 20401.

This Theorem, which we'll call *Theorem the third*, was first found out by Dr. *Halley*, and a notable Instance of its Use given by him in the *Philosophical Transactions* for making the Logarithm of 23 to 32 Places, by five Divisions performed with small Divisors; which could not be obtained according to the Methods first made use of, without indefatigable Pains and Labour, if at all; on account of the great Difficulty that would attend the managing such large Numbers.

Our Author's Series for this Purpose is (Page 357) $\overline{y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}}$, &c. the Investigation of which as he was pleased to conceal, induced me to inquire into it, as well to know the Truth of the Series, as to know whether this or that had the Advantage; because Dr. *Halley* informs us, when his was first published, that it converged quicker than any Theorem then made public, and in all Probability does so still. However that be, 'tis certain our Author's converges no faster than the second Theorem, as I found by the Investigation thereof, which may be as follows:

From the foregoing Doctrine, the Difference of the Logarithms of $\overline{z1-}$ and $\overline{z+1}$ is $m \times \frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5} + \frac{2}{7z^7}$, &c. which put equal to y, and the Logarithm of the Ratio of the Arithmetical Mean z, to the Geometrical Mean $\sqrt{zz-1}$ is $m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8}$, &c. *per* Theo-

rem

rem the second; for $z = \frac{1}{2}s$; whence $\frac{dd}{2ss} = \frac{1}{2zz}$.

Let A and B be the Logarithms of $z-1$ and $z+1$ respectively; then is $\frac{A+B}{2} + m \times \frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}$ the Logarithm of z; and if the latter Part of the Series expressing the said Logarithm of z be divided by the Series representing the Difference of the Logarithms of $z-1$ and $z+1$, the Quotient will exhibit the Series required, *viz.* $\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$, &c. as appears by the following Operation:

$$\frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5}\text{ \&c.}\Big)\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}\text{, \&c.}\Big(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}\text{, \&c.}$$

$$\frac{1}{2z^2} + \frac{1}{6z^4} + \frac{1}{10z^6}\text{, \&c.}$$

$$\frac{1}{12z^4} + \frac{1}{15z^6}\text{, \&c.}$$

$$\frac{1}{12z^4} + \frac{1}{36z^6}\text{, \&c.}$$

$$\frac{7}{180z^6}\text{, \&c.}$$

Now, because the Dividend is ever equal to the Divisor drawn into the Quotient of the Division; it follows, that $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$, &c. is equal to

$$m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6}\text{, \&c.}$$

But $\frac{A+B}{2} + m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6}$ &c. is the Logarithm of z; wherefore $\frac{A+B}{2} + y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$, &c. is the Logarithm of z. Q. E. I.

Note,

Note, I make the Auther's 5th Term $\frac{13}{25200}$, to be $\frac{1903}{2268000}$.

To illustrate this Theorem by an Example:
Let it be required to find the Logarithm of 101.
To the half Sum of the Logarithms of 100 and 102 = 2,0043000859

Add the Difference of the said Logarithms divided by 404, equal to	0,0000212876
And the Sum, *viz.* - - - - -	2,0043213735

is the Logarithm of 101 true to 9 Places of Figures: Whence it appears, that our Author's Series falls short of Dr. *Halley's*, in finding the Logarithm of the prime Number 101, three Places of Figures, by using only the first Terms of the Series; whereas, if two Terms in each were used, perhaps the Difference would have been considerably greater.

Note, This Series of our Author, deduced from Theorem the second, is in Effect Dr. *Halley's* too, but disguis'd by being thrown into a different Form; which, however, has its Use, as being very ready in Practice.

Having thus investigated several Theorems, whereby Tables of Logarithms, of any Form, may be constructed; it remains to shew how, from the Logarithm given, to find what Ratio it expresses.

The Logarithm of the Ratio of 1 to $1+x$ has been proved to be as $\overline{1+x}^{\frac{1}{n}} - 1 = \frac{1}{n} \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4}$, &c. n being any infinite Index whatsoever; whence, if L be put for the said Series, then $\overline{1+x}^{\frac{1}{n}} - 1 = L$; consequently $\overline{1+x}^{\frac{1}{n}} = 1 + L$, and $1 + x = \overline{1+L}^{n} = 1 + nL + \frac{1}{2}n^2L^2 + \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4$, &c.

AGAIN:

The Logarithm of the Ratio of 1 to $1-x$ has likewise been proved to be as $1 - \overline{1-x}^{\frac{1}{n}} = \frac{1}{n} \times \overline{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4}$, &c. $= L$, wherefore

Con-

$-\overline{1-x}^n = L$; and $\overline{1-x} = \overline{1-L}^n = 1 - nL + \frac{1}{2}n^2L^2 \quad n^3 L^3 + \frac{1}{24}n^4L^4$, &c.

Whence $1 \quad x = 1 + nL \pm \frac{1}{2}n^2L^2 \pm \frac{1}{6}n^3L^3$, &c. is a general Theorem for finding the Number from the Logarithm given of any Species or Form whatsoever; but in the Application of it to Practice we labour under a great Inconvenience, especially if the Numbers are large; that is to say, it converges so very slow, that it were much to be wished it could be contracted.

However, if L be the Logarithm of the Ratio of a the lesser Term, to b the greater, and either of them are given; then the other will be easily had, and expeditiously enough too:

$$\text{For } \frac{b}{a} \text{ or } \frac{a}{b} = 1 \pm nL + \frac{1}{2}n^2L^2 \pm \frac{1}{6}n^3L^3, \text{ \&c.}$$

Wherefore it follows, by the Help of a Table of Logarithms, that the corresponding Number to any Logarithm may be found, to as many Places of Figures as those Logarithms consist of: For, putting d equal to the Difference between the given Logarithm and the next less in the Table, then will the Number sought, *viz.* $N = a \times \overline{1 + nd + \frac{1}{2}n^2d^2 + \frac{1}{6}n^3d^3}$, &c. But if d be put equal to the Difference between the given Logarithm, and the next greater, then $N = b \times \overline{1 - nd + \frac{1}{2}n^2d^2 - \frac{1}{6}n^3d^3}$, &c. both which Series converge faster, as d is smaller.

But the first three Terms in each may be contracted into two, which is very useful, inasmuch as it saves the Trouble of raising n and d in the third Term to the second Power: For letting the first Term remain as it is, the other two are reduced to one, thus; make the second Term the Numerator of a Fraction, and Unity minus the third Term divided by the second is the Denominator.

$$\text{Whence } N = a \times \overline{1 + nd + \frac{1}{2}n^2d^2}$$

$$\text{becomes } N = a + \frac{ad}{\frac{1}{n} - \frac{1}{2}d};$$

$$\text{And } \quad N = b \times \overline{1 - nd + \frac{1}{2}n}$$

$$\text{becomes } N = b - \frac{bd}{\frac{1}{n} + \frac{1}{2}d}.$$

where-

wherefore $a+\frac{a\,d}{m-\frac{1}{2}d}$, or $b-\frac{b\,d}{m+\frac{1}{2}d}$, will be the Number anſwering to the given Logarithm; which, tho' it differs a little from the Truth, is ſufficient to find the Numbers, exact to as many Places as *Briggs*'s Logarithms conſiſt of, *viz.* 14, which are the largeſt Tables extant. Much after the ſame Method may the whole Series be contracted; by which Means each alternate Power of d will be exterminated; or, which is the ſame Thing, every two Terms in the Series will be reduced to one, making the Whole ſhorter by Half.

To illuſtrate theſe Conſtructions by an Example:

Let it be required to find the Number anſwering to the Logarithm 7,5713740282 in *Briggs*'s Form.

From the given Logarithm -	7,5713740282
Subtract the Log. of 372710000 the next neareſt - - - - - -	7,5713710453
The Remainder is equal to $d=$	,0000029829

And becauſe the Number 372710000 is leſs than the Number ſought, call it a, which, multiplied by ,0000029829, and the Product 1,111756659, &c. divided by $m-\frac{1}{2}d=$,4342929, &c. quotes 2559,92; which, added to 372710000, gives 372712559,92 for the Number ſought.

Thus, I preſume, the Doctrine of Logarithms has been ſufficiently exemplified, whether we conſider the Conſtruction of them for any given Numbers; or, on the contrary, the finding of the Numbers from the Logarithms given.

But, before I conclude, I ſhall give an Inſtance or two of the great Uſe of Logarithms in Arithmetical Calculations, and firſt in the purchaſing of Annuities.

If a be put for any Annuity, p for the preſent Value, r the Amount of One Pound for One Year at any Rate of Intereſt, and t for the Time or Number of Years the Annuity is to continue; then $p=\frac{\quad}{r-1}$, the Value of the Annuity.

EXAMPLE.

Let it be required to find the present Value of an Annuity of 60 *l. per Annum*, to continue 75 Years, at the Rate of 4 *per Cent. per Annum*.

Here $a=60$, $t=75$, and $r=1,04$. Now, in order to obtain the Answer, we must find the seventy-fifth Power of r, or of 1,04; that is, we must multiply 1,04 seventy-five Times into itself, which is exceeding tedious by the common Way, as any one may judge; but by the Logarithms 'tis done with the greatest Ease; for if 0,0170333 the Logarithm of 1,04 be multiplied by 75, the Product 1,2774975 will be the Logarithm of the seventy-fifth Power of 1,04; which being subtracted from 1,7781512, the Logarithm of a equal to 60, will leave 0,5006537 the Logarithm of 3,167041, which being subtracted from 60, and the Remainder divided by $r-1=,04$ will give $1420,824=1420$ *l.* 16 *s.* $5\frac{3}{4}$*d.* for the Value of the Annuity; and if 1420,824 be divided by 60, the Quotient will exhibit the Number of Years Purchase requisite to be given for any Annuity to continue 75 Years upon a good Security free of all Incumbrances, the Purchase being made at 4 *per Cent.*

Hence we see the Reason why the long Annuities purchased in the Year 1708, having about 75 Years to come, are valued in *Castain*'s Bill of Exchanges at $24\frac{1}{2}$ or 25 Years Purchase: For, tho' according to this Calculation, they are worth but a little more than 23 Years and a Half; yet, because in the public Funds 4 *per Cent.* is scarcely ever made of Money, and the Contingencies it is there subject to, which those Annuities, and other Government Securities, are not, makes them very justly worth $24\frac{1}{2}$ or 25 Years Purchase.

Likewise Questions relating to Annuities upon Lives, whether for one, two, or three, &c. are almost as easily estimated. For Instance; it may readily be found by Logarithms, that an Annuity for a Man of Thirty, to continue during his Life, is worth 11,61 Years Purchase, Interest 6 *per Cent.* but at 4 *per Cent.* 14,68. And as the Probabilities of Life's Continuance, and the Value thereof, are determined by an algebraical Process grounded upon the Rudiments of the Doctrine of Chances, and five Years Observations upon the Bills of Mortality of *Breslaw*, the Capital of *Silesia*; so there results that Truth and

Equity

Equity from the Operations, as ought to preside in all Contracts of this Nature. Whence it follows, that all other Methods, whose Resolution differs from this (especially if the Difference be much), may justly be deemed erroneous, and consequently prejudicial to one of the Parties concerned. Wherefore, to prevent Impositions thro' Ignorance, great Care should be taken; which Precaution, however unnecessary it may appear, 'tis presumed, it will be regarded, inasmuch as no one is willing to pay more Years Purchase than he has Chances for living; as, on the contrary, the Seller to receive less than his Due; which may possibly be by following the common Methods (where, for the most part, Regard is had neither to Age nor Interest) founded upon Caprice, Humour, or, if you please, Custom, the Contract being made, as they can agree, right or wrong; which Method of Procedure ought to be exploded, since so liable to Error, and the Consequences drawn therefrom so often wide of the Truth.

The other Instance which I shall give of the great Use of Logarithms is in the Case of *Sessa*, as related by Dr. *Wallis* in his *Opus Arithmeticum* from *Alsephad* (an *Arabian* Writer), in his Commentaries upon *Tograius*'s Verses; namely, that one *Sessa*, an *Indian*, having first found out the Game at *Chesse*, and shewed it to his Prince *Shehram*, the King, who was highly pleased with it, bid him ask what he would for the Reward of his Invention; whereupon he asked, That for the first little Square of the *Chesse* Board, he might have on Grain of Wheat given, for the second two, and so on, doubling continually, according to the Number of the Squares in the *Chesse* Board, which was 64. And when the King, who intended to give a very noble Reward, was much displeased, that he had asked so trifling an one, *Sessa* declared, that he would be contented with this small one. So the Reward he had fixed upon was ordered to be given him. But the King was quickly astonish'd, when he found, that this would rise to so vast a Quantity, that the whole Earth itself could not furnish out so much Wheat. But how great the Number of these Grains is, may be found by doubling one continually 63 Times, so that we may get the Number that comes in the last Place, and then one Time more to have the Sum of all; for the Double of the last Term less

by one is the Sum of all. Now this will be most expeditiously done by Logarithms, and accurately enough too for this Purpose: For if to the Logarithm of 1, which is 0, we add the Logarithm of 2 (which is 0,3010330) multiplied by 64, that is, 19,2659200, the absolute Number agreeing to this will be greater than 18446, 00000, 00000, 00000, and less than 18447, 00000, 00000, 00000.

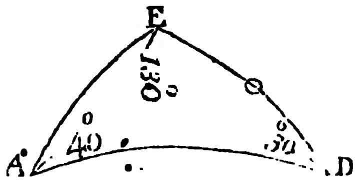
As I have had the revising of these Sheets, so it may be expected that I should give my Opinion concerning Mr. *Cunn* and our Author, in regard to Spherical Trigonometry; wherein the former accuses the latter, and several other eminent Authors, of having committed many Faults, and, in some Cases, of being mistaken, especially in the Solution of the 12th Case of Oblique Spherics; in which Mr. *Cunn* has intirely mistaken the Author's Meaning, as plainly appears by his Remark, where he constitutes a Triangle whose Sides are equal to the given Angles; whereas the Author means, that each Angle should first be changed into its Supplement, and then with the said Supplements another Triangle constituted, whose Angles, by the very Text of the 14th Proposition of his own Spherical Trigonometry, will be the Supplements of the Sides sought in the given Triangle; to which Proposition I refer the Reader. That this is the Sense of the Author, is very evident, if impartially attended to, and which I think could possibly have no other Meaning; and accordingly aver what is here advanced to be universally true; but, because I would not be misunderstood, shall illustrate the Truth thereof by a numerical Operation; which, to those who care not to trouble themselves with the Demonstration, may be sufficient; and, to others, some Satisfaction.

EXAMPLE.

Suppose, in the Oblique-angled Spherical Triangle ADE, there are given the Angles A, D, E, as *per* Figure, and the Side DE required.

Note, Write down the Supplements of the two Angles next the Side required first; and then the Operation my stand thus:

The



The Supplement of the Angle.	E = 50°	Sine Co. Ar.	0,115746
	D = 150	Sine Co. Ar.	0,301030
	A = 140	Sine 220°	9,937530
	Sum = 340	Sine 20	9,534052
	2)		
	Sum = 170		19,888358
			2)
½ Sum, minus the Supplement of the Angle.	E = 120		
	D = 20	- - - - - -	9,944179

Which last Figures 9,944179 give the Side of 61° 34′; and the Double thereof, *viz.* 123° 08′, subtracted from 180 Degrees, leaves for the Supplement 56° 52′, which is the Side of DE required.

The Rule which Mr. *Cunn* substitutes in the Room of our Author's, is also universal (but not new); and, consequently, when he says, Change one of the Angles adjacent to the Side sought into its Supplement, it is very just; though, by the way, I affirm, it is equally true, if the Angle opposite to the Side sought were changed into its Supplement (which perhaps is what has not yet been taken Notice of); only then, instead of having the Side sought directly, we should have its Complement to 180 Degrees, as in the preceding Example; but there is a Necessity of changing either one or all the Angles into their Supplements, though it is best to change only one, which let be either of those next the Side sought, no matter which; and the Side will be had directly without any Subduction, as will appear by the subsequent Operation.

EXAMPLE.

Let the Angle E be changed into its Supplement, and the Side DE sought; which Supplement, and the

other Angle adjacent to the Side fought, being written down firft, the Operation may be as follows:

Sup. of the Angle E=50	Sine Co. Ar.	—0,115746
The Angle { D=30	Sine Co. Ar.	—0,301030
A=40	Sine 10°	- 9,239670
Sum 120	Sine 30	- 9,698970
2		
½ Sum 60	Sum	19,355416
½ Sum minus { Sup. Angle E = 10 ; Angle D = 30	½ Sum	9,677708

Which half Sum 9,677708 gives the Sine of 28° 26′, and the Double thereof 56° 52′ is the Side DE fought, the fame as before, when all the Angles were changed into their Supplements.

Whence it is abundantly manifeft, that thofe two Methods of Operation, notwithftanding their Manner is fo different, agree precifely in Practice; and, confequently, we may conclude our Author's Rule to be right. Wherefore I wonder Mr. *Cunn* did not attend better to the Words of our Author's Rule, before he ventured to attack the Characters of fo many famous Trigonometrical Writers. But to remove the Imputation of the Charge againft thofe Authors who have deferved fo well of the Mathematics, and to juftify them to the World (for Juftice ought to have Place), it is, that I have ventured to give my Opinion, and point out where Mr. *Cunn* was miftaken: The Reafon of which is not eafily affigned, fince, to give him his Due, it could not be for want of Knowledge, tho', in this Cafe, I can't think it intirely owing to Inadvertence, inafmuch as it was a premeditated Thing; and I am loth to impute it to any contentious Inclinations of his, in difputing the Veracity of our Author's Rule, becaufe it did not appear with all that Plainnefs requifite to prevent carping by the Litigious: Wherefore, as I am in Sufpenfe how to determine, I fhall leave the Decifion thereof to better Judgments.

Indeed, Mr. *Heynes*'s Rule, which directs with the three Angles given to project a Triangle, as if they were Sides, is deficient, were it only on that very Account: For with the given Angles, in the preceding Example,

Example, it will be impossible to construct a Triangle, because 'tis requisite, that two Sides together, however taken, be greater than the third; whereas, in this Case, they will be less: But the Rule is not only deficient in that Respect, but really wrong: For tho' what Mr. *Heynes* asserts is just, *viz.* that the greatest Side in the supplemental Triangle is the Supplement of the greatest Angle in the other Triangle; yet, notwithstanding that, the Consequence drawn therefrom is false, and so the Solution only imaginary: For, with Submission, neither the Sides, nor their Supplements, in Mr. *Heynes*'s supplemental Triangle, are the Measures of the Sides sought. 'Tis true, when one of the Angles is a Right one, and the others both acute, then the said supplemental Triangle is that wanted to be constructed, as containing all the given Angles; and, consequently, the Sides appertaining thereto are the very Sides required: But then this is only one Instance out of the infinite Number of other Triangles that may be constructed, and which is not solved directly by the Triangle first projected neither; for the greatest Angle thereof must be changed into its Supplement, when the Side opposite to the Right Angle is required; and if the Right Angle still remains, and either one or both of the other given Angles are obtuse, the Solution is render'd more perplex'd: Wherefore there can be no general Solution given to any Triangle, by constituting a Triangle whose Sides are equal to the given Angles, except to that particular one which Mr. *Cunn* takes Notice of in his Remark, where each given Angle is the Measure of its opposite Side sought, and which therefore needs no Operation.

This I thought myself obliged to observe, in Justice to Mr. *Cunn*, who, we see, is not intirely to blame; as having just Reason to object against the Veracity of Mr. *Heynes*'s Rule, tho' not against the Rules of the other Authors by him nominated.

And here I can't but take Notice of some Gentlemen, who are so very fond of finding Fault, that, rather than you shall not be in the Wrong, they will wrest your own Meaning from you, and will not suffer an Error, tho' ever so minute, to pass, without proclaiming it to the Public, under Pretence of preventing their being impos'd upon; whereas, if the Truth were known, I fear it would appear to be the Vanity of their Hearts,

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an Over-fondneſs of being thought wiſer and more knowing than the reſt of Mankind; nay, I think, it appears plainly ſo, by their oppoſing the Works of Men greater than themſelves: But if, inſtead of comparing how far their finite Knowledge extend, or exceeded another Perſon's, they conſider'd how much there was they knew nothing of; as it would conduce to make them humble, ſo, I am of Opinion, it would contribute very much toward their leaving off that Manner of Writing. Beſides, as I take it, the Buſineſs of Writing is not ſo much to diſcover who has committed the moſt Faults, as to avoid them, and make greater Improvements.

But, what is the moſt to be wonder'd at, thoſe who [illegible] very ready in finding Fault, not without great Suſpicion, receive the beſt Part of their Knowledge from the Works of thoſe very Authors againſt whom they exclaim. The Reaſon that induces me to think ſo is this: Whilſt they are ſtudying an Author, in order to underſtand him, then it is, perhaps, they diſcover ſomething which he was pleaſed to omit, or thought fit to conceal, for which 'tis more than probable they take Care not to omit paying a profound Reſpect to their vainly-imagined ſuperior Geniuſes: And if, by Accident, an Error ſhould creep in (which is very poſſible, none being infallible), then, to be ſure, he muſt be egregiouſly miſtaken, and not underſtand what he was about: But, I ſay, this Diſquiſition into the Demerits of an Author would never have been made, had they underſtood the Subject beforehand; for, if otherwiſe, they muſt be of a ſad Cynical Temper, as well as have little elſe to do, to make it their Buſineſs to diſcover Faults, and at the ſame Time acknowledge not one ſingle Beauty; a very ungrateful Return for the Advantage they receive in the Peruſal.

Nor do they do the Public that Service they pretend to: For thoſe that are capable, and will be at the Trouble, of reading a Treatiſe upon a Subject without a Maſter, are as well able as themſelves to rectify what is amiſs; and as for thoſe who will not be at that Trouble, there is no Danger of their being led aſtray; ſince it is the ſame Thing to them, whether there be any Miſtakes, or not.

However, if, after all, there ſhould be a Neceſſity for an Admonition, why can't it be done with Candour and Humanity?

Humanity? And then, without doubt, an Author, out of Regard to Truth, which of all Things ought to be preferred, would be thankful: And to reprove otherwise, is to be ungenerous; because, whenever those Mistakes happen, as they are for the most part owing more to Inadvertency, than Want of Knowledge; so they should therefore be attributed to the Frailty of human Nature (to which we are all more or less subject), nothing being more common amongst all Professions, than the writing of one Thing for another.

If any think, by my interfering between our Author and Mr. *Cunn*, that I have run into the same Error, of which I accuse others in general of being guilty, let them please to consider that I have only writ in the Vindication of Gentlemen, who were first wrongfully accus'd, and in one Particular justify'd Mr. *Cunn*: For such an Occasion as this offering, I thought the Difference between them lay upon me to decide, lest I should be taxed with Partiality for not doing Justice, or with Ignorance in not determining an Affair which held some in Suspense to know who was in the right or wrong; for there could be no Possibility of making a Merit in adjusting a Thing of so easy a Nature; tho', perhaps, to conceive thoroughly the Reason of all the different Methods of Solution, may not be so easy neither.

But, to proceed: As for the Omissions our Author has made in not determining accurately when some of the Cases are ambiguous, and when not, I shall not quarrel with those who think him to blame; but, if I may be allowed to give my Opinion, I think they are determined for the most part, as well, or, at least, with more Ease, from the Construction of the Triangles, because it fixes an Idea of what one is about, by exhibiting a kind of an ocular Demonstration; and, consequently, prevents the laying of that Stress upon the Memory, as all those are obliged to who depend intirely upon Mr. *Cunn*'s Rules, which to Beginners is not very agreeable: Hence, who knows but that what our Author wrote relating to the ambiguous Cases, he thought sufficient? That is, that the Reader would not stop, for want of farther Explications, but with more Ease supply himself with what was wanting when he came to the Practice thereof, I mean the Construction of Triangles (for, after all, without the Knowledge of that, a Person will have

have but a mean Notion of this useful Branch of the Mathematics); and, if so, he ought in some measure to be excused, especially if to this we join the following Consideration, *viz.* that few or none ever learn Spherical Trigonometry, purely for the Sake of calculating Sides and Angles, to determine their Ambiguities; besides, what is ambiguous in Trigonometry, is very often not so in Geography and Astronomy, *&c.* for which the other is chiefly learnt.

For Instance: If we know the Latitude of *London*, and the Distance and Difference of Longitude between the said Place and *Rome*, notwithstanding there are two Sides, and the Angle opposite to one of them, given, the Case is not doubtful when we undertake to find the Latitude of *Rome*; unless it be not known whether it lies to the *Northward* or *Southward* of *London*; which however could not be determined by any Principles of Trigonometry. Likewise, in Astronomy, if the Latitude of the Place, the Sun's Declination and Azimuth, were given, the *Quæsitum* is not doubtful neither, unless the Sun's Declination exceeds the Latitude of the Place, and both are of the same Denomination, that is, both *North* or both *South*; in which Cases, because it is possible for the Sun to be upon the same Azimuth Circle, twice in the Forenoon, and upon another Azimuth Circle, twice in the Afternoon; it is doubtful, if by Circumstances, during the Observation, we can't discover which of the Times, whether the first, or last; but if those Times fall near each other, it will be quite impossible to distinguish which, and therefore ambiguous. Other Instances might be produced, but I believe these are sufficient to evince, that those nice Distinctions are not so necessary in Practice: If there be those who think otherwise, I shall not dispute it, but leave them to their Opinion without Interruption.

However, what with Mr. *Cunn*'s Rules for determining the ambiguous Cases (which are judiciously drawn up, as including all the Varieties possible), and the Corrections now made by restoring what was lost and corrupted, our Author's Treatise of Trigonometry, in respect to Theory, may perhaps appear compleat, even to the most scrupulous. And,

Here I thought to conclude; but, for the Sake of Novelty, and to illustrate the various Methods for solv-

ing the 12*th* Case of Oblique Spherics, where the three Angles are given to find either of the Sides, I shall beg Leave to give one Instance more, in order to shew how it may be perform'd after a new Manner, by the Help of the natural and logarithmical Versed Sines; which, if not intirely new, is not so publickly known as the preceding Methods; at least, I never saw any-where the Method of Operation, and therefore shall deliver a Rule for that Purpose, in the following Words:

RULE.

Having, according to the former Directions, chang'd one of the Angles next the Side sought into its Supplement; take the natural Versed Sine of the Difference of the said Supplement and the other adjacent Angle, and subtract it from the natural Versed Sine of the Angle opposite to the Side sought, and to the Logarithm of the Remainder add the Square of the Radius; then from the Sum subtract the logarithmical Sines of the above Supplement, and the same adjacent Angle; and the Remainder is the Logarithm of a Number, which will be the Versed Sine of the Side sought.

EXAMPLE.

Supplement	$<$ E $= 50^\circ$	
Angle	D $= 30$	
Natural V. Sine	Diff. $= 20 =$,06030	
	$<$ A $= 40 =$,23395	
	,17365	
The Log. of which Diff. ,17365 with the Square of Radius, is		29,239674
Sine of the Sup. of the $<$ E $50^\circ =$		9,884254
Add the Sine of the $<$ D $30 =$		9,698970
	Sum subtract	19,583224
	Remains	9,656450

Which Remainder 9,656450 gives the Logarithm Versed Sine of DE $56^\circ\ 52'$, agreeing exactly with the former Computations.

Note, If the said Remainder exceeds 10,000000, it implies that the Side sought is greater than a Quadrant; wherefore cancelling the Characteristic 10, look out for the Number answering the remaining Logarithm,

from

from which cut off the Left-hand Figure, or, which is the same Thing, abate the Radius (*viz.* Unity); and the Remainder will be the natural Sine of the Excess of the Side sought above a Quadrant.

As the natural and logarithmical Versed Sines are not so frequently met with in Books as the artificial Sines, 'tis possible, on that Account, this Rule may meet with some Objection; for which Reason, and not knowing whether it may be thought preferable to the foregoing Methods (tho' undoubtedly very easy in Practice), I have omitted its Demonstration; but have published the Rule, with some View of introducing the Use of the former Sines, which sometimes are preferable to the latter: For by the Help of the said Versed Sines, and the Reasoning used in obtaining this Rule, we necessarily come to the Knowledge of solving that Problem, where two Sides and the contained Angle are given, and the third Side required, at one Operation, very useful in Astronomy and Geography, especially in the latter; when the Latitudes and Longitudes of two Places are given to find their Distance asunder: But the Rule for performing it, and the Demonstration thereof, is also omitted for the Sake of Brevity.

However, 'tis easy to perceive, since Angles may be turned into Sides, that the present Rule includes the Solution of that useful Problem in Astronomy for finding the Sun's Azimuth, having the Latitude of the Place, the Sun's Altitude and Distance from the elevated Pole given; by which means the Variation of the Compass, of such Importance to Navigators, may be readily determined in any Part of the World.

An Example of which, comprehending the latter Part of the Rule (*viz.* when the Remainder exceeds 10,000000) is exhibited.

EXAMPLE.

Suppose on *June* the 30th, 1732, at *London*, in the Latitude of 51° 52′ N. it were required to find the Sun's true Azimuth, when his Altitude was 50° 00′, in the Afternoon. First,

From

		° ′	
From the Com. of the Altitude	-	40 00	
Sub. the Com. of the Latitude	-	38 28	
Natural V. Sine of the Difference	- -	1 32	,00035
Natural V. Sine of the Sun's Dist. from the Pole		67 54	,62377
			,62342

The Log. of which Difference ,62342 with the Square of the Radius, is — 29,794780

Cosine of the Latitude 51° 32′	- -	9,793831
Cosine of the Altitude 50 00	- -	9,808097
Sum subtract		19,601898
Remains		10,192882

Here the Remainder exceeds 10,000000; wherefore cancel the Characteristic 10, and the Number answering the remaining Logarithm is 1,5591; the Excess of which above Unity, *viz.* ,5591, gives the natural Sine of 34° 00′; whence the Sun's true Azimuth is North 124° 00′ West: At which Time, if the Sun's Magnetical Azimuth were North 110° 30′ West, the Variation of the Compass would be 13° 30′ West, as appears by the following Subtraction.

True Azimuth North,	124°	00′	West
Mag. Azimuth North,	110	30	West
Variation	13	30	West

N. B. If the Sun's Declination had been South, then the Versed Sine of the Sun's Distance from the elevated Pole would have been equal to Unity plus the natural Sine of the Sun's Declination; which in Practice creates no more Trouble than when the Declination is North, if so much; since it is at least as easy to take the natural Sine of an Arc, as to take the Versed Sine of its Complement to 90 Degrees; which Sines, and others, with their respective Logarithms, &c. may readily be had out of *Sherwin*'s Mathematical Tables.

F I N I S.

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